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Partially Ordered Sets - Basic Concepts

G. R. Brightwell and W. T. Trotter
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1.1 Chapter Overview

In this opening chapter, we provide a concise treatment of basic notation and terminology for partially ordered sets. This will include a discussion of covering relations, diagrams, chains and antichains, cartesian products, disjoint and lexicographic sums, and lattices. We encourage readers who are comfortable with the basics to skip ahead to Chapter 2, referring back to this introductory material as necessary. For those readers who know a bit but don't consider themselves as experts, give the chapter a quick read. It will be a helpful review. And for those readers who are new to partially ordered sets, we trust you will find this chapter and supporting exercises a good entry point.

1.2 Notation and Terminology

A binary relation P on a set X is called a *partial order* on X when it satisfies the following three conditions:

(i) If $x \in X$, then $(x, x) \in P$.
(ii) If $x, y \in X$, $(x, y) \in P$ and $(y, x) \in P$, then $x = y$.
(iii) If $x, y, z \in X$, $(x, y) \in P$ and $(y, z) \in P$, then $(x, z) \in P$.

These three conditions are called the *reflexive, antisymmetric* and *transitive* properties, respectively.

Example 1.2.1 *Let* $X = \{a, b, c, d, e, f\}$. *The following binary relations are partial orders on* X*:*

(i) $P_1 = \{(a, a), (b, b), (c, c), (d, d), (e, e), (f, f)\}$.
(ii) $P_2 = \{(a, a), (b, b), (c, c), (d, d), (e, e), (f, f), (a, b), (b, c), (a, c)\}$.

(iii) $P_3 = \{(a,a),(b,b),(c,c),(d,d),(e,e),(f,f),(a,c),(a,e),(a,f),$
$(b,c),(b,d),(b,e),(b,f),(d,e),(d,f),(e,f)\}$.

A *partially ordered set* or *poset* $\mathbf{P}$ is a pair (X,P) where X is a set and P is a partial order on X. We call X the *ground set* of the poset, and elements of the ground set X are also called *points*.

1.3 Alternate Notation for Partially Ordered Sets

A binary relation Q on a set X is called a *strict partial order* on X when it satisfies the following two properties.

(i) If $x \in X$, then $(x,x) \notin Q$.
(ii) If $x,y,z \in X$, $(x,y) \in Q$ and $(y,z) \in Q$, then $(x,z) \in Q$.

The first of these two properties is called the *irreflexive* property. Clearly, if Q is a strict partial order on X, then $P = Q \cup \{(x,x) : x \in X\}$ is a partial order on X, according to our original definition.

Example 1.3.1 *Let $X = \{a,b,c,d,e,f\}$. The following binary relations are strict partial orders on X. They correspond to the three partial orders given in Example 1.2.1 above.*

(i) $Q_1 = \emptyset$.
(ii) $Q_2 = \{(a,b),(b,c),(a,c)\}$.
(iii) $Q_3 = \{(a,c),(a,e),(a,f),(b,c),(b,d),(b,e),(b,f),(d,e),(d,f),(e,f)\}$.

When P is a partial order on X, many authors prefer to emphasize the order concept and write $x \le y$ in P when $(x,y) \in P$. Since a poset $\mathbf{P}$ is a pair (X,P), we can write $x \le y$ in $\mathbf{P}$ as a substitute for $x \le y$ in P. Of course, $y \ge x$ in P means the same as $x \le y$ in P, while the notations $x < y$ in P and $y > x$ in P mean $x \le y$ in P and $x \ne y$. Similarly, when Q is a strict partial order on X, we write $x < y$ in Q and $y > x$ in Q when $(x,y) \in Q$.

When the partial order P remains fixed throughout a discussion, it is convenient to abbreviate $x < y$ in P by just writing $x < y$. The same convention is used for strict partial orders.

Accordingly, a poset can be considered as either

(i) A pair $(X,\le)$ where $\le$ is a reflexive, antisymmetric and transitive relation on X, or
(ii) A pair $(X,<)$ where $<$ is an irreflexive and transitive relation on X.

When discussing more than one partial order on a ground set X, it is useful to distinguish between the partial order and the ground set and refer to a poset as a pair (X, P). On the other hand, in many settings, the partial order is fixed throughout the discussion. In this case, many authors will refer to a poset with a single symbol, such as P or Q, so they will write, for example: Let P be a poset on n points, and let $x \in P$. This statement means that the ground set of the poset has n points and that the element x belongs to this ground set.

The same conventions will be used for graphs. In particular, while we formally consider a graph $\mathbf{G}$ as a pair (V, E) where V is a vertex set and E is an edge set, we will just say G is a graph when the vertex set and edge set remained fixed throughout the discussion.

1.4 Comparability, Incomparability and Cover Graphs

Let $\mathbf{P} = (X, P)$ be a poset. If x and y are points in X (not necessarily distinct), and either $x \leq y$ in P or $y \leq x$ in P, we say x and y are *comparable* in $\mathbf{P}$; else they are *incomparable* in $\mathbf{P}$. Note that each element of X is comparable with itself. We will write $x \| y$ in P when x and y are incomparable in P. Also, we will let $\operatorname{Inc}(\mathbf{P}) = \{(x, y) \in X \times X : x \| y \text{ in } P\}$. Note that $\operatorname{Inc}(\mathbf{P})$ is symmetric and irreflexive. On the other hand, there is no standard notation for comparable pairs.

Associated with a poset $\mathbf{P} = (X, P)$ is a *comparability* graph $\mathbf{G}$ which has X as its vertex set with distinct points (vertices) adjacent in $\mathbf{G}$ if and only if they are comparable in $\mathbf{P}$. We will say more about these graphs in Chapter 9.99. The *incomparability* graph of a poset is defined analogously.

Again, let $\mathbf{P} = (X, P)$ be a poset and let x and y be distinct points in X. We say x is *covered* by y in $\mathbf{P}$ (also y *covers* x in $\mathbf{P}$) when $x < y$ in P, and there is no point $z \in X$ for which both $x < z$ in P and $z < y$ in P. We associate with the poset $\mathbf{P}$ a *cover* graph having X as its vertex set with distinct points adjacent when one of them covers the other in $\mathbf{P}$.

The concept of a cover graph provides a convenient scheme for drawing diagrams of posets. A drawing of the cover graph of $\mathbf{P}$ in the euclidean plane equipped with a standard cartesian coordinate system (first coordinate axis horizontal and second coordinate axis vertical) is called a *poset diagram* (also a *Hasse diagram*) when the second coordinate of y is larger than the second coordinate of x whenever y covers x in $\mathbf{P}$. It is customary to use straight line drawings for poset diagrams, although this is not really essential. However, when non-straight line drawings are employed, it is still customary to use

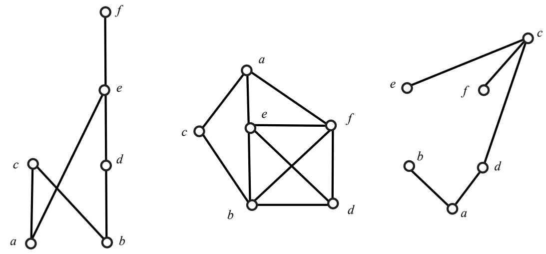

Fig. 1.1. A Poset and its Comparability and Incomparability Graphs

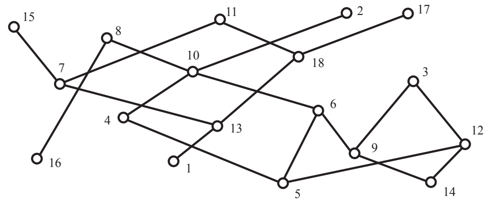

Fig. 1.2. A Partially Ordered Set

piece-wise linear edges that flow "downwards" from y to x whenever y covers x in $\mathbf{P}$.

In Figure 1.4, we show on the left a diagram for the poset (X, P_3), where $X = \{a, b, c, d, e, f\}$ and P_3 is the partial order given in Example 1.2.1. In the center of Figure 1.4, we show the comparability graph of this poset and on the right its incomparability graph.

1.5 More Examples of Posets

There are many different settings in which partial orders arise in a natural way. Here are two such examples.

Example 1.5.1 *Let X be any family of sets and set $P = \{(A, B) \in X \times X : A \subseteq B\}$.*

Example 1.5.2 *Let X be any set of positive integers and set $P = \{(m, n) \in X \times X : m$ divides n without remainder$\}$.*

However, in general, we will be concerned with posets for which there is no "natural" explanation for the order. For example, consider the poset with ground set $X = \{1, 2, \ldots, 18\}$ whose diagram is shown in Figure 1.5.

1.6 Basic Concepts

When $\mathbf{P} = (X, P)$ is a poset and Y is a nonempty subset of X, the *restriction* of P to Y, denoted by $P(Y)$ (also denoted $P|_Y$), is a partial order on Y and we call $(Y, P(Y))$ a *subposet* of (X, P). We also say that $(Y, P(Y))$ is the subposet of $\mathbf{P}$ *generated* (also, *determined*, or *induced*) by Y. When the poset remains fixed in a discussion, we will just refer to the subposet Y. When Y is a proper subset of X, we will speak about the subposet determined by $X - Y$ as the subposet obtained by *removing* Y. Also, when $Y = \{x\}$, we will just talk about the subposet obtained by removing the point x.

A poset $\mathbf{P} = (X, P)$ is called a *chain* if x is comparable to y in P, for all $x, y \in X$. When (X, P) is a chain, we call P a *linear order* (also, *total order*) on X. Througout this monograph, we will use the symbols $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ to denote the reals, rationals, integers and positive integers, respectively. Each of these number systems comes equipped with a total order $\leq$, which we call the *natural* order. Accordingly, in discussing a poset whose ground set is labelled by elements of $\mathbb{R}$, some care should be exercised when using the $<$ and $\leq$ notations. For example, for the poset shown in Figure 1.5, we have $10 < 2$ and $7 \| 18$, while $2 < 7 < 10 < 18$ in the natural order on $\mathbb{R}$.

A nonempty subset $Y \subseteq X$ is called a *chain* if the subposet $(Y, P(Y))$ is a chain. A one-element chain is said to be *trivial*, while a chain of two or more points is *nontrivial*.

Dually, a poset $\mathbf{P} = (X, P)$ is called an *antichain* when x and y are incomparable in P whenever $x, y \in X$ and $x \neq y$. A nonempty subset $Y \subseteq X$ is called an *antichain* if the subposet $(Y, P(Y))$ is an antichain. A one-element antichain is said to be *trivial* while antichains of two or more points are *nontrivial*. Note that a one-element subset of X is both a chain and an antichain.

Throughout this monograph, we will use the short form $[n]$ to denote the n-element set $\{1, 2, \ldots, n\}$. However, it is customary to use the bold-face notation $\mathbf{n}$ to denote the n-element chain $0 < 1 < 2 < \cdots < n - 1$. On the other hand, there is no standard notation for an n-element antichain.

A point $x \in X$ is called a *maximal* point of $\mathbf{P}$ if there is no point $y \in X$ with $x < y$ in P. We denote the set of all maximal points of a poset $\mathbf{P}$ by $\text{Max}(\mathbf{P})$. Since we use the notation $\mathbf{P} = (X, P)$ for a poset, the set of maximal elements can also be denoted by $\text{Max}(X, P)$ or just $\text{Max}(P)$. This convention will be used for all set valued and integer valued functions of posets.

Similarly, a point $x \in X$ is called a *minimal* point of $\mathbf{P}$ if there is no point

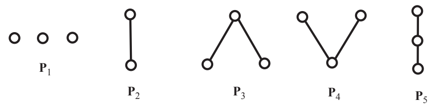

Fig. 1.3. Unlabeled Posets on 5 Points

$y \in X$ with $x > y$ in P. We denote the set of all minimal points of a poset $\mathbf{P}$ by $\mathrm{Min}(\mathbf{P})$, $\mathrm{Min}(X, P)$ or $\mathrm{Min}(P)$.

An element $x \in X$ is called a *greatest* (or *maximum*) point of $\mathbf{P}$ if $y \leq x$ in P for every $y \in X$. Similarly, x is called a *least* (or *minimum*) point of $\mathbf{P}$ if $y \geq x$ in P for every $y \in X$. Note that if X is a finite set, then $\mathrm{Max}(\mathbf{P})$ and $\mathrm{Min}(\mathbf{P})$ are always non-empty. On the other hand, posets may not have greatest points, and they may not have least points. Also, when X is an infinite set, it may happen that one or both of $\mathrm{Max}(\mathbf{P})$ and $\mathrm{Min}(\mathbf{P})$ is empty. For example, the set $\mathbb{N}$ of positive integers has a least element (namely, the positive integer 1), but the set of maximal elements is empty.

The set of all chains in a poset $\mathbf{P} = (X, P)$ is partially ordered by set inclusion and the maximal elements in this poset are called *maximal chains*. A chain C is *maximum* if no other chain contains more points than C. *Maximal* and *maximum antichains* are defined analogously. Both $\mathrm{Max}(P)$ and $\mathrm{Min}(P)$ are maximal antichains; however, it may happen that neither is a maximum antichain.

The *height* of a poset (X, P) is the number of points in a maximum chain, and the *width* is the number of points in a maximum antichain. It is easy to see that the poset shown in Figure 1.5 has height 5, but it is not so easy to see that its width is 7. We will revisit this issue in Chapter 9.99.

Let $\mathbf{P} = (X, P)$ and $\mathbf{Q} = (Y, Q)$ be posets and let $f : X \xrightarrow[\text{onto}]{1\text{–}1} Y$ be an bijection between the two ground sets. We say f is an *isomorphism* from $\mathbf{P}$ to $\mathbf{Q}$ if $x_1 \leq x_2$ in P if and only if $f(x_1) \leq f(x_2)$ in Q. When there is an isomorphism from $\mathbf{P}$ to $\mathbf{Q}$, we say that $\mathbf{P}$ is *isomorphic* to $\mathbf{Q}$ and write $\mathbf{P} \cong \mathbf{Q}$. An isomorphism from $\mathbf{P}$ to $\mathbf{P}$ is called an *automorphism* of $\mathbf{P}$, and an isomorphism from $\mathbf{P}$ to a subposet of $\mathbf{Q}$ is called an *embedding* of $\mathbf{P}$ in $\mathbf{Q}$.

The notion of isomorphism leads in a natural way to the concept of an *unlabeled* poset. For example, in Figure 1.6, we show the five unlabeled posets on three points. There are respectively 1, 6, 3, 3 and 6 ways to assign labels from $\{1, 2, 3\}$ so there are 19 labeled posets with ground set $\{1, 2, 3\}$.

In many settings, we will not distinguish between isomorphic posets, and we will say that a poset $\mathbf{P}$ is *contained* in $\mathbf{Q}$ (also $\mathbf{Q}$ *contains* $\mathbf{P}$) when there is an embedding of $\mathbf{P}$ in $\mathbf{Q}$. Also, we will frequently abuse notation and write $\mathbf{P} = \mathbf{Q}$ when $\mathbf{P}$ and $\mathbf{Q}$ are isomorphic.

The *dual* of a partial order P on a set X is denoted by P^d and is defined by $P^d = \{(y, x) : (x, y) \in P\}$. Note that the dual of a partial order is again a partial order. The *dual* of the poset $\mathbf{P} = (X, P)$ is denoted by $\mathbf{P}^d$ and is defined by $\mathbf{P}^d = (X, P^d)$. A poset $\mathbf{P}$ is *self-dual* if $\mathbf{P} = \mathbf{P}^d$ (of course, here we really mean $\mathbf{P} \cong \mathbf{P}^d$).

A poset $\mathbf{P} = (X, P)$ is *connected* if its comparability graph G is connected, and when $\mathbf{P}$ is not connected, a subposet $\mathbf{Q} = (Y, Q)$ is called a *component* of $\mathbf{P}$ when the point set Y of $\mathbf{Q}$ induces a component in G. A one-point component is *trivial*, and is also called a *loose* point or an *isolated* point. Components of two or more points are *nontrivial*.

For the poset $\mathbf{P} = (X, P)$ shown in Figure 1.5, note that:

(i) $\mathrm{Max}(\mathbf{P}) = \{2, 3, 8, 11, 15, 17\}$.
(ii) $\mathrm{Min}(\mathbf{P}) = \{1, 5, 14, 16\}$.
(iii) $\{1, 7, 13, 15\}$ is a maximal chain.
(iv) $\{6, 8, 9, 10, 14\}$ is a maximum chain, so $\mathrm{height}(\mathbf{P}) = 5$.
(v) $\{2, 3, 7, 8, 18\}$ is a maximal antichain.
(vi) $\{4, 6, 11, 12, 15, 16, 17\}$ is a maximum antichain, so $\mathrm{width}(\mathbf{P}) = 7$.
(vii) $\mathbf{P}$ is disconnected and has two components.

A poset $\mathbf{P} = (X, P)$ is *finite* if the ground set X is finite. In view of our combinatorial emphasis, we will concentrate almost exclusively on finite posets in this monograph. Exceptions will include subposets of the real number system $\mathbb{R}$, and in particular, the rationals $\mathbb{Q}$, the integers $\mathbb{Z}$ and the positive integers $\mathbb{N}$. Also, in Chapter 9.99, we will consider a special class of posets where the ground set is countably infinite.

1.7 Linear Extensions

Let P and Q be partial orders on a set X. We say Q is an extension of P if $Q \subseteq P$; and we say an extension Q of P is a *linear extension* if Q is a linear order on X. The following proposition is trivial for finite sets but involves set theoretic issues for infinite posets.

Proposition 1.7.1 *If P is a partial order on a set X, and $(x, y) \in \mathrm{Inc}(\mathbf{P})$, then there exists a linear extension L of P with (y, x) in L.*

The proof of Proposition 1.7.1 follows easily from the following somewhat more technical result.

Proposition 1.7.2 *If P is a partial order on a set X, $x, y \in X$ and $(x, y) \in \mathrm{Inc}(\mathbf{P})$, then the transitive closure of $P \cup \{(y, x)\}$ is a partial order on X.*

Example 1.7.3 *The poset* $\mathbf{P}$ *shown in Figure 1.4 has eleven linear extensions. These are displayed vertically in the following table:*

L_1	L_2	L_3	L_4	L_5	L_6	L_7	L_8	L_9	L_{10}	L_{11}
f	f	f	c	f	f	f	c	f	f	c
e	e	c	f	e	e	c	f	e	c	f
d	c	e	e	d	c	e	e	c	e	e
c	d	d	d	c	d	d	d	a	a	a
b	b	b	b	a	a	a	a	d	d	d
a	a	a	a	b	b	b	b	b	b	b

Here is an important observation, one that we will revisit in Chapter 9.99.

Proposition 1.7.4 *Let $\mathbf{P} = (X, P)$ be a poset. Then the family $\mathcal{E}$ of all linear extensions of P is nonempty and $P = \bigcap \mathcal{E}$.*

Proof If P is a linear order, then $\mathcal{E} = \{P\}$ and $\bigcap \mathcal{E} = P$. If P is not a linear order, then for each $(x, y) \in \mathrm{Inc}(\mathbf{P})$, there exists a linear extension $L(x, y)$ of P with $(y, x) \in L$. Thus $(x, y) \notin L$. Since $\mathrm{Inc}(\mathbf{P})$ is symmetric, it follows that there is no $(x, y) \in \mathrm{Inc}(\mathbf{P})$ for which $(x, y) \in \bigcap \mathcal{E}$. Thus $\bigcap \mathcal{E} = P$. $\square$

When $\mathbf{P} = (X, P)$ is a poset, L is a linear extension of P and $|X| = n$, there is a natural map $h_L : X \to [n]$ defined by setting $h_L(x) = |\{y \in X : y \leq x \text{ in } L\}|$. The value $h_L(x)$ is called the *height* of x in L. To illustrate this elementary, in Example 9.99, the height of e in L_1 is 5, while the height of d in L_6 is 3.

1.8 Extensions of a Partial Order

Let $\mathbf{P} = (X, P)$ be a poset and let $\mathcal{P}$ denote the family of all extensions of P (all extensions, not just the linear extensions). Given a set $S \subset \mathrm{Inc}(\mathbf{P})$, it is useful to have a test to determine whether there is some $Q \in \mathcal{P}$ which contains S. From Proposition 1.7.1, we know the answer is yes when S consists of a single ordered incomparable pair.

An *alternating cycle* in $\mathbf{P}$ is a sequence $\{(x_i, y_i) : 1 \leq i \leq k\}$ of ordered pairs from $\mathrm{Inc}(\mathbf{P})$ with $y_i \leq x_{i+1}$ in P (cyclically†) for $i = 1, 2, \ldots, k$. The

† By the statement $y_i \leq x_{i+1}$ in P (cyclically) for $i = 1, 2, \ldots, k$, we mean the same thing as $y_i \leq x_{i+1}$ in P for $i = 1, 2, \ldots, k-1$ and $y_k \leq x_1$ in P.

integer k in this definition is called the *length* of the cycle. An alternating cycle is *strict* if $y_i \le x_j$ in P if and only if $j = i + 1$ (cyclically) for $i, j = 1, 2, \ldots, k$.

Example 1.8.1 *In Figure 1.5, the set* $S = \{(16, 11), (13, 8), (9, 13), (18, 9)\}$ *is an alternating cycle of length 4. It is not strict. However, the set* $S' = \{(16, 7), (3, 8), (18, 3)\}$ *is a strict alternating cycle of length 3.*

It is important to note that when $S = \{(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)$ is a strict alternating cycle, then $\{x_1, x_2, \ldots, x_k\}$ and $\{y_1, y_2, \ldots, y_k\}$ are k-element antichains in $\mathbf{P}$.

The following elementary result is due to Trotter and Moore [99].

Proposition 1.8.2 *Let* $\mathbf{P} = (X, P)$ *be a poset and let* $S \subset \mathrm{Inc}(\mathbf{P})$. *Then the following statements are equivalent:*

(i) The transitive closure of $P \cup S$ is not a partial order on X.
(ii) S contains an alternating cycle.
(iii) S contains a strict alternating cycle.

Proof The fact that Statement 3 implies Statement 2 is immediate. Statement 2 implies Statement 1 since the transitive closure of $P \cup S$ would contain both (x_1, y_1) and (y_1, x_1). We now show that Statement 1 implies Statement 3. Let Q be the transitive closure of $P \cup S$. Since Q is both reflexive and transitive, the fact that it is not a partial order means that it must violate the antisymmetric property. This means that there is a sequence $(u_1, u_2, \ldots, u_m)$ of points from X (not necessarily distinct) so that $(u_i, u_{i+1}) \in P \cup S$ for each $i = 1, 2, \ldots, n$ (cyclically). Of all such sequences, choose one for which the value of m is minimum. Since P is antisymmetric, we may assume that $(u_1, u_2) \in S$. Then set $x_1 = u_1$ and $y_1 = u_2$. Now suppose we have defined a pair (x_i, y_i) with $y_i = u_j$ with $2 \le j \le m$. If $(u_j, u_{j+1}) \in S$, set $x_{i+1} = y_i = u_j$ and $y_{i+1} = u_{j+1}$. If $(u_j, u_{j+1}) \in P$, then $(u_{j+1}, u_{j+2}) \in S$. In this case, set $x_{i+1} = u_{j+1}$ and $y_{i+1} = u_{j+2}$. Clearly, this construction yields a strict alternating cycle contained in S. $\square$

1.9 Algebraic Operations and Lattices

Let $\mathbf{P} = (X, P)$ and $\mathbf{Q} = (Y, Q)$ be posets. The *cartesian product* of the two posets, denoted $\mathbf{P} \times \mathbf{Q}$, is the poset $(X \times Y, R)$, where $(x_1, y_1) \le (x_2, y_2)$ in R if and only if $x_1 \le x_2$ in P and $y_1 \le y_2$ in Q. We illustrate this definition in Figure 1.9.

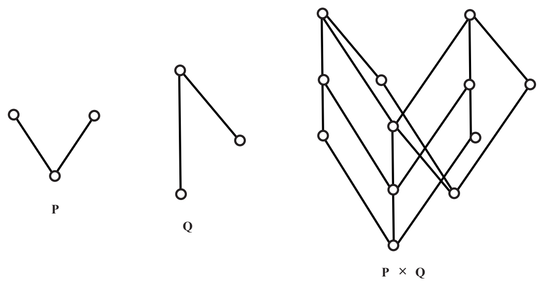

Fig. 1.4. A Cartesian Product

Let $\mathbf{P} = (X, P)$ and $\mathbf{Q} = (Y, Q)$ be posets with $X \cap Y = \emptyset$. The *sum* of the two posets, denoted $\mathbf{P} + \mathbf{Q}$, is the poset $(X \cup Y, P \cup Q)$. To emphasize the requirement that the ground sets be disjoint, this operation is also called *disjoint sum.* Of course, when $\mathbf{P}$ and $\mathbf{Q}$ have overlapping ground sets, we can still talk about the disjoint sum $\mathbf{P} + \mathbf{Q}$ by first taking isomorphic copies of the two posets and artificially making their respective ground sets disjoint. Here we are again taking advantage of the convention that isomorphic posets can be considered equal.

With these comments in mind, the following basic proposition is immediate.

Proposition 1.9.1 *Let* $\mathbf{P}$*,* $\mathbf{Q}$ *and* $\mathbf{R}$ *be posets. Then*

(i) $\mathbf{P} + \mathbf{Q} = \mathbf{Q} + \mathbf{P}$.
(ii) $\mathbf{P} + (\mathbf{Q} + \mathbf{R}) = (\mathbf{P} + \mathbf{Q}) + \mathbf{R}$.
(iii) $\mathbf{P} \times \mathbf{Q} = \mathbf{Q} \times \mathbf{P}$.
(iv) $\mathbf{P} \times (\mathbf{Q} \times \mathbf{R}) = (\mathbf{P} \times \mathbf{Q}) \times \mathbf{R}$.
(v) $\mathbf{P} \times (\mathbf{Q} + \mathbf{R}) = (\mathbf{P} \times \mathbf{Q}) + (\mathbf{P} \times \mathbf{R})$.

In view of the preceding proposition, it is natural to denote the cartesian product of n copies of a poset $\mathbf{P}$ by $\mathbf{P}^n$. As will be clear, the special case when $\mathbf{P} = \mathbf{2}$ is particularly important. In Figure 1.9, we show a diagram for $\mathbf{2}^4$.

More generally, the poset $\mathbb{R}^t$ is just the set of all t-tuples (vectors) of real numbers with $(x_1, x_2, \ldots, x_t) \leq (y_1, y_2, \ldots, y_t)$ in $\mathbb{R}^t$ if and only if $x_i \leq y_i$ for all $i = 1, 2, \ldots, t$. So an embedding of a poset $\mathbf{P} = (X, P)$ in $\mathbb{R}^t$ assigns to each $x \in X$ a vector $(x_1, x_2, \ldots, x_t)$ in $\mathbb{R}^t$ so that $x \leq y$ in P if and only if $x_i \leq y_i$ in $\mathbb{R}$ for each $i = 1, 2, \ldots, t$.

Now let $\mathbf{P} = (X, P)$ be a poset and let $\mathcal{F} = \{\mathbf{Q}_x : x \in X\}$ be a family of posets indexed by the elements in the ground set of $\mathbf{P}$, and for each

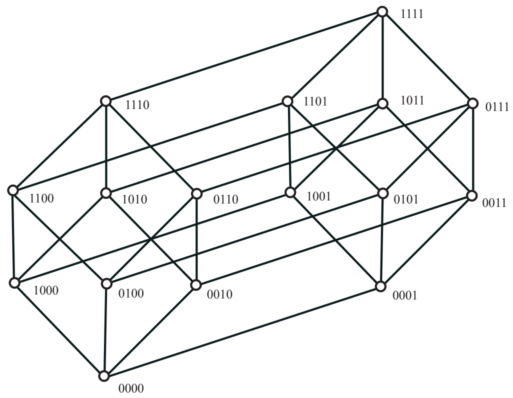

Fig. 1.5. The Poset $\mathbf{2}^4$

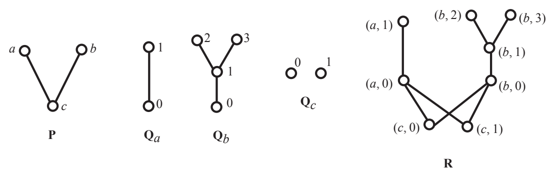

Fig. 1.6. A Lexicographic Sum

$x \in X$, let $\mathbf{Q}_x = (Y_x, Q_x)$. We define the *lexicographic sum of $\mathcal{F}$ over* $\mathbf{P}$ as the poset $\mathbf{R} = (Z, R)$ where $Z = \{(x, y_x) : x \in X, y_x \in Y_x\}$ and $R = \big\{\big((x_1, y_1), (x_2, y_2)\big) \in Z \times Z :$ either $x_1 < x_2$ or both $x_1 = x_2$ and $(y_1, y_2) \in Q_{x_1}\big\}$. We illustrate this defnition with the family of posets in Figure 1.9.

A lexicographic sum is *trivial* when either (1) $|X| = 1$; or (2) $|Y_x| = 1$, for every $x \in X$. If Condition 1 holds, then the lexicographic sum (Z, R) is isomorphic to (Y_x, Q_x), where x is the unique element of X; and if Condition 2 holds, then (Z, R) is isomorphic to (X, P). A poset (Z, R) is *decomposable* when it is (isomorphic to) a non-trivial lexicographic sum; else, it is *indecomposable*. Note that the only indecomposable disconnected poset is a 2-element antichain.

Let $\mathbf{P} = (X, P)$ be a poset and let $S \subseteq X$. An element $b \in X$ is called an *upper bound* for S if $s \leq b$ in P, for every $s \in S$. An upper bound b for S is the *least upper bound* of S, abbreviated l.u.b.(S), provided $b \leq b'$ in P for every upper bound b' of S. *Lower bounds* and *greatest lower bounds* are defined analogously. A poset $\mathbf{P} = (X, P)$ is called a *lattice* if every nonempty subset $S \subseteq X$ has both a least upper bound and greatest lower bound. When $\mathbf{P} = (X, P)$ is a lattice, we have natural functions, $\vee$ and $\wedge$, from $X \times X$ to X defined by setting

$$x \vee y = \mathrm{l.\,u.\,b.}\{x, y\} \quad \text{and} \quad x \wedge y = \mathrm{g.\,l.\,b.}\{x, y\}.$$

When $x, y \in X$, $x \vee y$ is called the *join* of x and y, while $x \wedge y$ is called the *meet* of x and y.

Lattices always have least and greatest elements, and when the lattice contains more than one point, the least element is traditionally called *zero*, denoted 0, and the greatest element is *one*, denoted 1.

When X is a set, the family of all subsets of X forms a lattice and is called a *subset lattice*. Note that $\mathbf{2}^n$ is isomorphic to the family of subsets of $\{1, 2, \ldots, n\}$ ordered by inclusion, and the elements can then be denoted using subsets of $[n]$ or as 0–1 strings (bit strings) of length n. The cover graph of $\mathbf{2}^n$ is called an *n-cube* and is frequently denoted $\mathbf{Q}_n$.

When $\mathbf{P} = (X, P)$ and $\mathbf{Q} = (Y, Q)$ are posets, a function $f : X \to Y$ is *order preserving* (respectively, *order reversing*) if $x_1 \leq x_2$ in P implies $f(x_1) \leq f(x_2)$ (respectively, $f(x_1) \geq f(x_2)$) in Q for all $x_1, x_2 \in X$. Let $\mathbf{P} = (X, P)$ and $\mathbf{Q} = (Y, Q)$ be posets. Then we let $\mathbf{Q}^{\mathbf{P}}$ denote the poset whose ground set consists of all order preserving functions from $\mathbf{P}$ to $\mathbf{Q}$. The partial order is then defined by setting $f_1 \leq f_2$ in $\mathbf{Q}^{\mathbf{P}}$ if and only if $f_1(x) \leq f_2(x)$ in Q for every $x \in X$.

Proposition 1.9.2 *Let* $\mathbf{P}$, $\mathbf{Q}$ *and* $\mathbf{R}$ *be posets. Then*

$$\mathbf{Q}^{\mathbf{P}+\mathbf{R}} = \mathbf{Q}^{\mathbf{P}} \times \mathbf{Q}^{\mathbf{R}}.$$

A lattice is *distributive* if it satisfies the following two properties:

(i) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge y)$.
(ii) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee y)$.

When $\mathbf{P} = (X, P)$ is a poset, it is easy to see that the poset $\mathbf{2}^{\mathbf{P}}$ is a distributive lattice. For the three point poset $\mathbf{P}$ shown in the left part of Figure 1.9, there are five order preserving functions from $\mathbf{P}$ to $\mathbf{2}$: f_1, f_2, f_3, f_4 and f_5, where

(i) $f_1(a) = f_1(b) = f_1(c) = 1$.
(ii) $f_2(a) = f_2(b) = 1$ and $f_2(c) = 0$.
(iii) $f_3(a) = 1$ and $f_3(b) = f_3(c) = 0$.
(iv) $f_4(b) = 1$ and $f_4(b) = f_4(c) = 0$.
(v) $f_5(a) = f_5(b) = f_5(c) = 0$.

In fact, the preceding example characterizes distributive lattices.

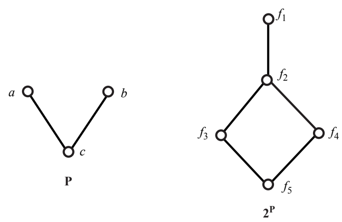

Fig. 1.7. A Distributive Lattice

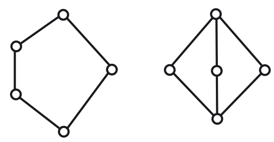

Fig. 1.8. Two Non-Distributive Lattices

Proposition 1.9.3 *A finite lattice* $\mathbf{L}$ *is distributive if and only if there is a finite poset* $\mathbf{P}$ *so that* $\mathbf{L} = \mathbf{2}^{\mathbf{P}}$.

When $\mathbf{P} = (X, P)$ is a poset, a subset $D \subseteq$ is called a *down set* in $\mathbf{P}$ if $y \in D$ whenever $x \in D$ and $x \leq y$ in P. *Up sets* are defined analogously. It is easy to see that the characterization of distributive lattices given in Proposition 1.9.3 can be restated in terms of down sets.

Proposition 1.9.4 *A finite lattice* $\mathbf{L}$ *is distributive if and only if there is a finite poset* $\mathbf{P}$ *so that* $\mathbf{L}$ *is isomorphic to the family of all down sets of* $\mathbf{P}$ *partially ordered by inclusion.*

There is still another way to characterize distributive lattices, one that is illustrative of a number of results in this monograph. First, note that neither of the lattices shown in Figure 1.9 are distributive.

Proposition 1.9.5 *A lattice* $\mathbf{L}$ *is distributive if and only if it does not contain either of the lattices shown in Figure 1.9 as sublattices.*

Exercises

1.1 For $n = 4, 5$, how many posets are there with ground-set $X = \{1, 2, 3, \ldots, n\}$?
1.2 For $n = 4, 5$, how many unlabelled posets on n points?
1.3 Show that every finite poset has a minimal element.

1.4 Let $(X, \leq)$ be any poset. Show that there is some collection $\mathcal{F}$ of sets such that $(X, \leq)$ is isomorphic to $(\mathcal{F}, \subseteq)$.

1.5 Let $(X, \leq)$ be any poset. Show that there is a set S of positive integers and a bijection $f : X \xrightarrow[\text{onto}]{1\text{–}1} S$ so that $x \leq y$ if and only if $f(x)$ divides $f(y)$ without remainder.

1.6 Let $\mathcal{F} = \{[a_x, b_x] : x \in X\}$ be a family of closed intervals of $\mathbb{R}$. Define a strict partial order P on $\mathcal{F}$ by $[a_x, b_x] < [a_y, b_y]$ in P if and only if $b_x < a_y$ in $\mathbb{R}$. Show that the poset $(\mathcal{F}, P)$ does not contain a subposet isomorphic to $\mathbf{2} + \mathbf{2}$.

1.7 Show that the set of all partitions of a set Y, with order-relation given by refinement, forms a lattice. Show however that, for $|Y| \geq 3$, this lattice is not distributive.

1.8 Suppose that $\mathbf{L}_1$ and $\mathbf{L}_2$ are distributive lattices. Show that the cartesian product $\mathbf{L}_1 \times \mathbf{L}_2$ is also a distributive lattice.

1.9 Suppose that $\mathbf{L}$ is a finite lattice satisfying either of the two requirements given in the definition of a distributive lattice, then it also satisfies the other requirement.

1.10 For each positive integer m, construct a poset with exactly m linear extensions.

1.11 For which numbers $m \leq 10$ does there exist a poset with exactly m extensions? (Don't forget that a poset is an extension of itself.)

1.12 Prove Proposition 1.7.2 by showing that the transitive closure of $P \cup \{(x, y)\}$ is a partial order when x and y are incomparable points in a poset $\mathbf{P} = (X, P)$.

1.13 For any set X with at least two elements, let $\mathcal{B}(X)$ be the set of all partial orders on X. The $\mathcal{B}$ is partially ordered by inclusion, and the maximal elements of the resulting posets are the linear orders on X. Attach a new element to this poset and make it greater than all other elements. Explain why the resulting poset is not a lattice.

1.14 For the poset described in the preceding exercise, show that if $P < Q$, then the restriction to all partial orders R on X with $P \subseteq R \subseteq Q$ is a lattice. Is it distributive?

1.15 For each $i = 0, 1, \ldots, n-1$, let $\mathbf{Q}_i$ be a two element antichain. Then let $\mathbf{R}$ be the lexicographic sum of the family $\mathcal{F} = \{\mathbf{Q}_i : 0 \leq i \leq n-1\}$ over $\mathbf{n}$. How many linear extensions does $\mathbf{R}$ have?

1.10 Notes and References

Issues which need to be addressed.

(i) Should there be an abstract? WTT: Yes, and one has been provided. It is called a "Chapter Overview."
(ii) Should there be references? WTT: Yes! Those now lised are very incomplete.
(iii) What about some historical remarks? WTT: Yes!

References

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