4

Extremal Set Theory

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4.1 Chapter Overview

Extremal set theory is the label broadly applied to combinatorial problems for the subset lattice: the family of all subsets of a finite set, partially ordered by inclusion. We include in this chapter a representative sampling of classic results, including Sperner's theorem and its application to the Littlewood-Offord problem and the Erdős-Ko-Rado theorem. However, we quickly shift the focus to more modern work on hamiltonian cycles and paths, partitions into intervals and k-crossing families.

4.2 The Subset Lattice and Sperner's Theorem

For a positive integer n, we let $\mathbf{2}^n$ denote the subset lattice consisting of all subsets of $[n] = \{1, 2, \ldots, n\}$ ordered by inclusion. Of course, we may also consider $(2)^n$ as the set of all 0–1 strings of length n with partial order

$$\mathbf{a} = (a_1, a_2, \ldots, a_n) \le \mathbf{b} = (b_1, b_2, \ldots, b_n)$$

if and only if $a_i \le b_i$ for each $i = 1, 2, \ldots, n$. We illustrate this with a diagram for $\mathbf{2}^4$ in Figure 4.2.

Some elementary properties of the poset $\mathbf{2}^n$ are:

(i) The height is $n+1$ and all maximal chains have exactly $n+1$ points.

(ii) The size of the poset $\mathbf{2}^t$ is 2^n and the elements are partitioned into ranks (antichains) $A_0, A_1, \ldots, A_n$ with $|A_i| = \binom{n}{i}$ for each $i = 0, 1, \ldots, t$.

(iii) The maximum size of a rank in the subset lattice occurs in the middle, i.e. if $s = \lfloor n/2 \rfloor$, then the largest binomial coefficient in the sequence $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$ is $\binom{n}{s}$. Note that when n is odd, there are two ranks of maximum size, but when n is even, there is only one.

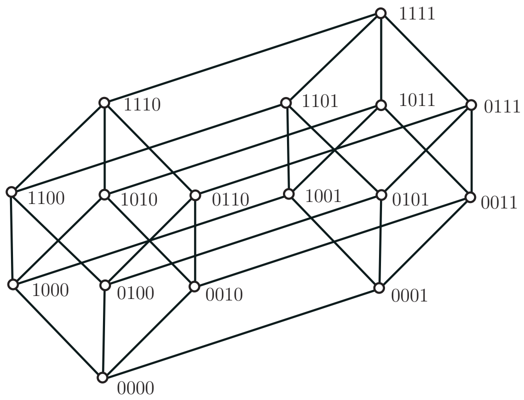

Fig. 4.1. The Subset Lattice $\mathbf{2}^4$

4.2.1 Sperner's Theorem

For the width of the subset lattice, we have the following classic result due to Sperner, although the proof we give here is due (independently) to Lubell, Yamamoto and Meshalkin [99].

Theorem 4.2.1 (Sperner) *For each* $n \geq 1$, *the width of the subset lattice* $\mathbf{2}^n$ *is the maximum size of a rank, i.e.,*

$$\text{width}(\mathbf{2^n}) = \binom{\mathbf{t}}{\lfloor \frac{\mathbf{n}}{\mathbf{2}} \rfloor}$$

Proof The width of the poset $\mathbf{2}^n$ is at least $C(n, \lfloor n/2 \rfloor)$ since the set of all $\lfloor n/2 \rfloor$-element subsets of $[n]$ is an antichain. We now show that the width of $\mathbf{2}^n$ is at most $C(n, \lfloor n/2 \rfloor)$.

Let w be the width of $\mathbf{2}^n$ and let $\mathcal{F}$ be an antichain of size w in this poset. For each non-negative integer k, let $\mathcal{F}_k = \{S \in \mathcal{F} : |S| = k\}$. Note that for each set $S \in \mathcal{F}_k$, the number of maximal chains passing through S is exactly $n!(n-k)!$. Since no maximal chain can pass through two distinct sets from $\mathcal{F}$, it follows that

$$w = \sum_{k=0}^{n} |\mathcal{F}_k| k!(n-k)! \leq n!$$

And we conclude that

$$\begin{aligned}\frac{w}{\binom{n}{\lfloor n/2\rfloor}} &= \frac{\sum_{k=0}^{n} |\mathcal{F}_k|}{\binom{n}{\lfloor n/2\rfloor}}\\ &\leq \frac{\sum_{k=0}^{n} |\mathcal{F}_k|}{\binom{n}{k}}\\ &= \frac{\sum_{k=0}^{t} |\mathcal{F}_k|(k!(n-k)!}{n!}\\ &\leq 1\end{aligned}$$

So that $w \leq \binom{n}{\lfloor n/2\rfloor}$. □

The reader should note that the proof as presented actually establishes a somewhat stronger result, which we state here for emphasis. When $\mathcal{F}$ is an antichain in $\mathbf{2}^n$, and $\mathcal{F}_k = \{S \in \mathcal{F} : |S| = k\}$, then

$$\sum_{k=0}^{n} \frac{|\mathcal{F}_k}{\binom{n}{k}} \leq 1$$

4.2.2 Littlewood/Offord Problem for Reals

Here is a lovely little problem, and in the discussion, we go back and forth between treating elements of $\mathbf{2}^n$ as 0–1 vectors of length n and as subsets of $[n]$.

We consider the following problem, first posed by Littlewood and Offord [99]. Let $x_1, x_2, \ldots, x_n$ be (not necessarily distinct) real numbers with $|x_i| \geq 1$ for each $i = 1, 2, \ldots, n$. What is the maximum size of a family $\mathcal{F}$ of 0–1 vectors from $\mathbf{2}^n$ satisfying the following property: For every pair S, T from $\mathcal{F}$,

$$|\sum_{i=1}^{n} s_i x_i - \sum_{i=1}^{n} t_i x_i| < 1.$$

Erdős noted that Sperner's theorem can quickly be applied to provide the answer: $\binom{n}{\lfloor n/2\rfloor}$. For the lower bound, take all $x_i's$ to be 1. Then let $\mathcal{F}$ consist of all 0–1 vectors of length n with exactly $\lfloor n/2 \rfloor$ 1's. Then all sums of the form $\sum_{i=1}^{n} s_i x_i$ are exactly equal to $\lfloor n/2/rfloor$. For the upper bound, first note that we may assume that all $x_i > 0$. for if $x_i < 0$, simply replace x_i by $-x_i$ and then take the family $\mathcal{F}' = \{S\Delta\{i\} : S \in \mathcal{F}\}$.

Now observe that $\mathcal{F}$ must be an antichain, for if $S, T \in \mathcal{F}$ and S is a proper subset of T, then

$$\sum_{i=1}^{n} t_i x_i - \sum_{i=1}^{n} s_i x_i \geq 1$$

But now consider the same extremal problem but this time with x_1, $x_2, \ldots, x_n$ allowed to be complex numbers. Again setting all $x_i = 1$, we have the same lower bound, but at least as it has been formulated here, the simple argument given by Erdős for the upper bound does not seem to apply. To remedy this, we first find a different proof of Sperner's theorem.

4.2.3 Symmetric Chain Partitions

A poset P is said to be *ranked* if all maximal chains have the same cardinality. When a poset is ranked, then there is a partition $X = A_1 \cup A_2 \cup \ldots A_h$ so that every maximal chain consists of exactly one point from each A_i. We call this partition its *partition into ranks*.

A ranked poset is said to be *sperner* if the width of the poset is just the maximum cardinality of a rank. So using this terminology, Sperner's theorem is just the assertion that the subset lattice is Sperner.

Let P be a ranked poset of height h and let A_1, $A_2, \ldots, A_h$ be the ranks of P. A chain C in P is called a *symmetric chain* if there exists an integer s so that C contains exactly one point from each rank $A_s, A_{s+1}, \ldots, A_{h+1-s}$. Intuitively, a symmetric chain is (1) balanced about the middle of the poset and (2) dense in the sense that it is not possible to insert a point in between two consecutive points in C.

The following proposition is self evident.

Proposition 4.2.2 *If a ranked poset has a partition into symmetric chains, then it is a Sperner poset. In fact, its width is just the size of the middle rank(s).*

So an alternative proof of Sperner's theorem is provided by the following result, due independently to Katona [99] and somebodyelse [99].

Theorem 4.2.3 *For each $n \geq 1$, the subset lattice $\mathbf{2}^n$ has a symmetric chain partition.*

In fact, we prove a much stronger result.

Theorem 4.2.4 *If P and Q are ranked posets and each has a symmetric chain partition, then $P \times Q$ is ranked and has a symmetric chain partition.*

Note that Theorem **??** follows immediately from Theorem **??** since $\mathbf{2}^n$ is just the cartesian product of n copies of the two-element chain $\mathbf{2}$, and this has a trivial symmetric chain partition.

The argument for Theorem **??** begins with a technical lemma.

Lemma 4.2.5 *Let m and n be positive integers. Then the cartesian product $\mathbf{m} \times \mathbf{n}$ has a symmetric chain partition.*

Proof The point set of $\mathbf{m}\times\mathbf{n}$ is just $\{(i,j) : 0 \le i < m, 0 \le j < n\}$. Without loss of generality $m \le n$, so that the width of $\mathbf{m} \times \mathbf{n}$ is m. Then for each $i = 0, 1, \ldots, m-1$, let

$$C_i = \{(i,0), (i,1), \ldots, (i, n-1-i), (i+1, n-1-i), \ldots, (m-1, n-1-i)\}.$$

Then the family $\{C_1, C_2, \ldots, C_m\}$ is a symmetric chain partition of $\mathbf{m} \times \mathbf{n}$. □

We are now ready for the proof of Theorem **??**. It is easy to see that if P is ranked and has height h_1 and Q is ranked and has height h_2, then $P \times Q$ is ranked and has height $h_1 + h_2 - 1$. Now suppose that P and Q have symmetric chain partitions. Let C be a chain from the partition of P and let D be a chain from the partition of Q. Then apply the preceding lemma to obtain a partition of the product $C \times D$. What results is a saturated partition of $P \times Q$.

4.2.4 Littlewood/Offord Problem for Complex Numbers

Let n be a positive integer and let $t = \binom{n}{\lfloor n/2 \rfloor}$. We pause to take a closer look at the details of a symmetric chain partition of $\mathbf{2}^n$. Regardless of how such a partition is constructed, it always results in a partition $\mathbb{P}$ of the family of all subsets of $[n]$ so that:

(i) There are t families in $\mathbb{P}$.
(ii) For each i, let t_i count the number of families of $\mathbb{P}$ which contain exactly i sets. Then $t_{n+1} = 1$, $t_{n-1} = n - 1$, $t_{n-3} = \binom{n}{3} - t_1 - t_2$, etc. In particular, the value of t_i depends only on n and not on the symmetric chain partition.
(iii) Each family $\mathcal{F}$ in $\mathbb{P}$ is a symmetric chain.

Accordingly, we say that a partition $\mathbb{P}$ of the family of all subsets is *symmetric* when it satisfies the first two properties listed above. It is important to note that we do *not* require that a symmetric partition also satisfy the third property. In particular, we do not require the sets form a chain, and in fact, we make no assumption about the sizes of the sets in the family.

Example 4.2.6 *Here is symmetric partition of* $\mathbf{2}^4$*.*

$$\begin{aligned}
\mathcal{F}_1 &= \{\{4\}, \{1,2\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}\} \\
\mathcal{F}_2 &= \{\emptyset, \{1,3\}, \{1,4\}\} \\
\mathcal{F}_3 &= \{\{2\}, \{3\}, \{1,2,3,4\}\} \\
\mathcal{F}_4 &= \{\{1\}, \{2,3\}, \{2,3,4\}\} \\
\mathcal{F}_5 &= \{\{2,4\}\} \\
\mathcal{F}_6 &= \{\{3,4\}\}
\end{aligned}$$

Now fix a vector $X = (x_1, x_2, \ldots, x_n)$ of complex numbers. We say that a family $\mathcal{S}$ of sets (0–1 vectors) from $\mathbf{2}^n$ is *sparse* if

$$|\sum_{i=1}^{n} s_i x_i - \sum_{i=1}^{n} t_i x_i| \geq 1$$

for every distinct pair $S = (s_1, s_2, \ldots, s_n)$ and $T = (t_1, t_2, \ldots, t_n)$ from $\mathcal{S}$. With this background in place, we can now present the solution of the Littlewood-Offord problem in the complex number system. The result is due independently to Kleitman [99] and Katona [99].

Lemma 4.2.7 *There exists a symmetric partition of* $\mathbf{2}$ *into sparse families.*

Proof We proceed by induction on n, observing first that the result hold trivially for $n = 1$. Now suppose it holds when $n \leq m$ and consider the case that $n = m + 1$. First take a symmetric partition $\mathbb{P}$ of $\mathbf{2}^m$ into sparse families for the vector $(x_1, x_2, \ldots, x_m)$. For each family $\mathcal{F}$ in $\mathbb{P}$, we will form two new families in $\mathbb{Q}$ by identifying a particular set S from $\mathcal{F}$ and then taking: $\mathcal{F}_1 = \mathcal{F} \cup \{S \cup \{x_{m+1}\}$ and $\mathcal{F}_2 = \{T \cup \{x_{m+1}\} : T \in \mathcal{F}, T \neq S\}$. Note that regardless of how S is chosen, the family $\mathcal{F}_2$ is sparse, since it is a translate of a sparse family. The challenge is to choose S so that $\mathcal{F}_1$ is also sparse.

But this choice is easy. For each set $S = (s_1, s_2, \ldots, s_n) \in \mathcal{F}$, we consider of the complex number $\sum_{i=1}^{n} s_i x_1$ on x_{m+1}. choose x_i so that this projection is maximum. We illustrate this choice in Figure 4.2.4 where the correct set is S_2.

STUFF NEEDED HERE!! □

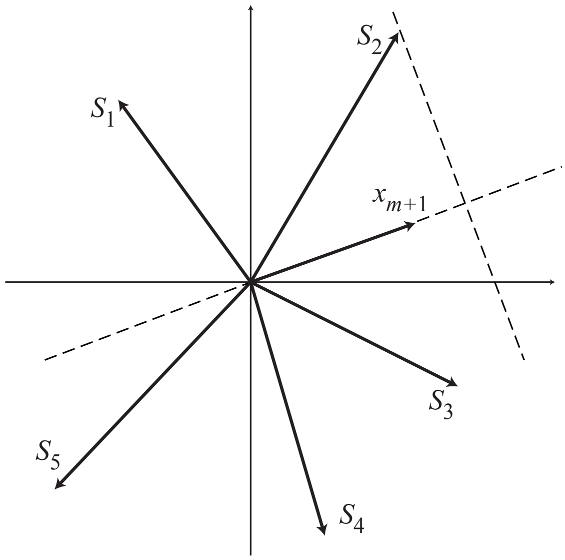

Fig. 4.2. Littlewood-Offord for Complex Numbers

4.3 The Erdős-Ko-Rado Theorem

Sperner's theorem tells us the width of the subset lattice. It is interesting to answer how large an antichain $\mathcal{F}$ in the subset lattice can be if $\mathcal{F}$ is required to satisfy additional properties. The argument given for Sperner's theorem also works in the following case.

Proposition 4.3.1 *Let n and r be positive integerrs with $2r \leq n$. Then let $\mathcal{F}$ be an antichain in $\mathcal{F}$ with $|A| \leq r$ for every $A \in \mathcal{F}$. Then $|\mathcal{F}| \leq r$.*

But here is a twist that requires a new idea. The result is known as the Erdős-Ko-Rado theorem, but the proof we present is due to Katona [**?**].

Theorem 4.3.2 (Erdős-Ko-Rado) *Let n and r be positive integerrs with $2r \leq n$. Then let $\mathcal{F}$ be a an antichain in $\mathcal{F}$ with $|A| \leq r$ for every $A \in \mathcal{F}$. If $A \cap B \neq \emptyset$ for every $A, B \in \mathcal{F}$, then $|\mathcal{F}| \leq C(n, r)$.*

Proof Consider all possible arrangements of the integers in $[n]$ at n equally spaced points around a circle. There are $n!$ arrangements, one for each permutation σ of $[n]$. Now let $\mathcal{B}$ denote the set of all pairs (A, σ) where A is a set from $\mathcal{F}$ and the elements of A occur as a block (in the cyclic sense) in the arrangment σ. If $|A| = s \leq r$, then there are $ns!(n-s)!$ pairs in $\mathcal{B}$

having A as the first coordinate. On the other hand, for each arrangement σ, there are clearly at most r pairs from $\mathcal{B}$ having σ as second coordinate. If there are f_s sets in $\mathcal{F}$ having cardinality s, so that $|\mathcal{F}| = \sum_{s=1}^{r} f_s$, then

$$\sum_{i=1}^{r} f_s ns!(n-s)! \leq rn!$$

so that

$$\sum_{i=1}^{r} \frac{s}{r} \frac{(s-1)!(n-s)!}{(n-1)!} \leq 1.$$

This show that $|\mathcal{F}| \leq C(n-1, r-1)$. □

The inequality in Theorem **??** is clearly best possible, since we may take $\mathcal{F}$ as the family of all r-element subsets of $[n]$ which contain the element 1. Also note that in the spirit of the LYM approach to Sperner's theorem, the proof actually yields a somewhat stronger result.

4.4 Antichains with Limits on Crossings

In this section, we consider an intriguing extremal problem posed to us by Piotr Micek, and in the intervening months, a number of researchers have made contributions to our understanding of the problem. However, the major result to date is Theorem **??** and this is due solely to B. Walczyk.

For a positive integer w, the elements of $\mathbb{Z}^w$ are just vectors of the form $x = (x_1, x_2, \ldots, x_w)$ with each x_i an integer. Considering $\mathbb{Z}^w$ as a cartesian product of w copies of $\mathbb{Z}$, the natural order on $\mathbb{Z}^w$ is defined by setting $x \leq y$ when $x_i \leq y_i$ in $\mathbb{Z}$ for each $i = 1, 2, \ldots, w$. When considering a When considering a sequence $x(1), x(2), \ldots, x(m)$ of vectors from $\mathbb{Z}^w$, the notation $x_\alpha(i)$ will mean coordinate α of vector $x(i)$.

A set A of elements in $\mathbb{Z}$ is an antichain if and only for each distinct pair $x, y \in A$, there are coordinates i and j for which $x_i \geq 1 + y_i$ and $y_j \geq 1 + x_j$. More generally, when k is a positive integer, we say that x and y are k-crossing when there are coordinates i and j so that $x_i \geq k + y_i$ and $y_j \geq k + x_j$. With this terminology, we have the following natural extremal problem.

For positive integers k and w, what is the maximum size $f(k, w)$ of an antichain A in $\mathbb{Z}^w$ so that there is no k-crossing pair in A. Trivially, $f(k, w) = 1$ if either $k = 1$ or $w = 1$ so we are really interested in $f(k, w)$ when both k and w are at least 2.

Proposition 4.4.1 *$f(k, 2) = k$ for all $k \geq 1$.*

Proof For the lower bound, consider the vectors $\{(i, k-1-i) : 0 \leq i \leq k-1\}$. For the lower bound, let A be an antichain in $\mathbb{Z}^2$ which does not contain a k-crossing pair. Then for each distinct pair $x, y \in A$, $x_1 \neq y_1$. Furthermore, if $x_1 < y_1$, then $x_2 > y_1$. It follows then if $|A| \geq k+1$, then A has a k-crossing pair. □

For arbitrary values of w, we have the following two basic results.

Proposition 4.4.2 *For all $k, w \geq 1$, $f(k,w) \geq k^{w-1}$.*

Proof The bound holds when $w = 1$, so we assume $w \geq 2$. Now let A be the following set:

$$A = \{(x_1, x_2, \ldots, x_w) : 0 \leq x_i \leq k-1 \text{for all } i-1, 2, \ldots, w-1; x_1+x_2+\cdots+x_w = 0\}$$

Clearly $|A| = k^{w-1}$ and A has no k-crossing pair. □

Proposition 4.4.3 *For all $k, w \geq 1$, $f(k,w) \leq k^w$.*

Proof Let A be an antichain in $\mathbb{Z}^w$ which does not contain a k-crossing pair. For each vector $x \in A$, let $\sigma(x)$ be the vector from $\mathbf{k}^w$ so that coordinate i of $\sigma(x)$ is j when $x_i \cong j \mod k$. If $\sigma(x) = \sigma(y)$ for distinct elements x and y of A and we choose coordinates i and j for which $x_i > y_i$ and $y_j > x_j$, then these two coordinates witness that x and y are k-crossing. We conclude that σ is an injection. This completes the proof. □

In the exercises, we describe two alternate constructions of antichains in $\mathbb{Z}^w$ which also establish the lower bound $f(k,w) \geq k^{w-1}$. Armed with not much more than the facts that (1) the lower bound is tight when $w = 1$ and $w = 2$ holds, and (2) we have not been able to build bigger antichains when $w \geq 3$, we conjecture that $f(k,w) = k^{w-1}$ for all $k, w \geq 1$. To lend credibility to the conjecture, we settle the case $w = 3$. As a warm up, the reader is encouraged to put the monograph down and take a few minutes time to show that $f(2,3) = 4$. Then come back to reading.

Theorem 4.4.4 (Walczyk) *The maximum size $f(k,3)$ of an antichain in $\mathbb{Z}^3$ which does not contain a k-crossing pair is k^2.*

Proof As noted previously, we assume $k \geq 2$. Let A be an antichain in $\mathbb{Z}^3$ which does not contain a k-crossing pair, with $|A| = f(k,3)$. We will