ON-LINE DIMENSION FOR POSETS EXCLUDING TWO LONG INCOMPARABLE CHAINS

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Abstract. For a positive integer k, let $\mathbf{k}+\mathbf{k}$ denote the poset consisting of two disjoint k-element chains, with all points of one chain incomparable with all points of the other. Bosek, Krawczyk and Szczypka showed that for each $k \geq 1$, there exists a constant c_k so that First Fit will use at most $c_k w^2$ chains in partitioning a poset P of width at most w, provided the poset excludes $\mathbf{k}+\mathbf{k}$ as a subposet. This result played a key role in the recent proof by Bosek and Krawczyk that $O(w^{16 \log w})$ chains are sufficient to partition on-line a poset of width w into chains. This result was the first improvement in Kierstead's exponential bound: $(5^w-1)/4$ in nearly 30 years. Subsequently, Joret and Milans improved the Bosek-Krawczyk-Szczypka bound for the performance of First Fit to $8(k-1)^2 w$, which in turn yields the modest improvement to $O(w^{14 \log w})$ for the general on-line chain partitioning result. In this paper, we show that this class of posets admits a notion of on-line dimension. Specifically, we show that when k and w are positive integers, there exists an integer $t = t(k,w)$ and an on-line algorithm that will construct an on-line realizer of size t for any poset P having width at most w, provided that the poset excludes $\mathbf{k}+\mathbf{k}$ as a subposet.

1. Introduction

Recall that a family $\mathcal{R} = \{L_1, L_2, \dots, L_t\}$ of linear extensions of a partially ordered set (poset) P is called a *realizer* of P if $x < y$ in P if and only if $x < y$ in L_i for each $i = 1, 2, \dots, t$. The *dimension* of P is then defined as the least t for which P has a realizer of cardinality t. We refer the reader to Trotter's monograph [20] and survey article [21] for extensive background material on dimension and combinatorial problems for finite posets.

When P and Q are posets and there is no subposet of P that is isomorphic to Q, we will simply say that P *excludes* Q. In this paper, we will consider the on-line version of dimension introduced in [12], and we will consider classes of posets that exclude two long incomparable chains. To be more precise, for a positive integer k, let $\mathbf{k}+\mathbf{k}$ denote the poset consisting of two disjoint k-element chains, with all points of one chain incomparable with all points of the other. Then our principal result will be the following theorem.

Theorem 1.1. *Let k and w be positive integers. Then there exists an integer $t = t(k,w)$ and an on-line algorithm that will construct an on-line realizer of size t for any poset P having width at most w and excluding $\mathbf{k}+\mathbf{k}$.*

The remainder of this paper is organized as follows. In the next section, we provide a brief sketch of results that motivate our line of research. In Section 2,

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we develop some key properties of posets that exclude $\mathbf{k}+\mathbf{k}$, and in Section 4, we provide the proof of our principal theorem. Finally, in Section 5, we highlight some open problems.

2. Background Material

As is customary in discussions of on-line algorithms, we consider the problem as a two-person game: OL-Dim$(\mathcal{P}, n, t)$ where $\mathcal{P}$ is a class of posets and n and t are positive integers. One person, called a *Builder*, constructs a poset P from $\mathcal{P}$ one point at a time, while the second player, called an *Assigner*, builds a realizer of this poset in an on-line manner. If the ground set is $\{x_1, x_2, \dots, x_n\}$, then at round i, Builder will have described the subposet of P_i induced by the elements $\{x_1, x_2, \dots, x_i\}$ and Assigner will have determined a family $\mathcal{R}_i = \{L_1^i, L_2^i, \dots, L_t^i\}$ of linear extensions forming a realizer of P_i. Both constructions proceed in an on-line manner, i.e., when $i > 1$, Builder need only list the elements of $\{x_1, x_2, \dots, x_{i-1}\}$ that are, respectively, less than x_i, greater than x_i and incomparable with x_i in P_i. Subsequently, for each $j = 1, 2, \dots, t$, Assigner reveals how each L_j^{i-1} from $\mathcal{R}_{i-1}$ will be extended to form a linear extension L_j^i, while maintaining the property that $\mathcal{R}_i$ must be a realizer of P_i.

The game ends, with Builder declared the winner, if at some round i with $2 \le i \le n$, Builder presents the required information for P_i but Assigner cannot extend the extensions from $\mathcal{R}_{i-1}$ to maintain $\mathcal{R}_i$ as a realizer of P_i. Assigner wins if Builder does not, i.e., after all n rounds are completed, Assigner has a realizer $\mathcal{R}_n$ of the final poset $P = P_n$.

We say the on-line dimension of a class $\mathcal{P}$ of posets is infinite if for every t, there is some n so that Builder has a winning strategy for the game OL-Dim$(\mathcal{P}, n, t)$. Here, we are particularly interested in classes $\mathcal{C}$ for which the on-line dimension is finite, i.e., there is some t for which Assigner has a winning strategy for the game OL-Dim$(\mathcal{P}, n, t)$ for all n. The least such t is called the *on-line dimension* of the class, and when we are unable to settle the exact value, we would at least like to provide reasonable upper bounds.

2.1. **On-Line Chain Partitions.** Any discussion of on-line dimension can't go very far without mentioning the companion problem of constructing an on-line chain partition of a poset. Here we have a two person game OL-ChainPart$(\mathcal{P}, n, s)$ where $\mathcal{P}$ is a class of posets and n and s are positive integers. In this game, *Builder* constructs a poset one point at a time and *Partitioner* constructs a chain partition in an on-line manner. At round i, with $1 \le i \le n$, Builder describes the subposet P_i induced by the elements of $\{x_1, x_2, \dots, x_i\}$. *Partitioner* currently has a chain partition $\mathcal{C}_{i-1} = \{C_1^{i-1}, C_2^{i-1}, \dots, C_s^{i-1}\}$ (initialized by setting all chains in $\mathcal{C}_0$ to be empty), and this partition is updated to $\mathcal{C}_i$ by choosing an appropriate j_0 with $1 \le j_0 \le s$ and setting $C_{j_0}^i = \{x_i\} \cup C_{j_0}^{i-1}$. Of course, for all $j = 1, 2, \dots, s$, with $j \neq j_0$, $C_j^i = C_j^{i-1}$.

We say that class $\mathcal{P}$ can be partitioned on-line into s chains when Partioner has a winning strategy for the game OL-ChainPart$(\mathcal{P}, n, s)$, for every n.

Historically, the following theorem of Kierstead [10] played a very important role in motivating research on on-line problems for posets.

Theorem 2.1. *The class of all posets of width at most w can be partitioned on-line into $(5^w - 1)/4$ chains.*

From below, an argument due to Szemerédi shows that any on-line algorithm can be forced to use at least $w(w+1)/2$ chains to partition on-line posets of width w into chains. In [1], this argument is presented, together with an improved lower bound of $(1-o(1))w^2$. The upper bound has proved equally resilient, but quite recently, Bosek and Krawczyk [3] made a significant advancement by proving the first subexponential bound for on-line chain partitioning.

Theorem 2.2. *The class of all posets of width at most w can be partitioned on-line into $w^{16 \log w}$ chains.*

2.2. **Excluding Two Incomparable Chains.** Posets that exclude $\mathbf{2}+\mathbf{2}$ as subposets are just the interval orders [6], and for this class, we have the following result [15].

Theorem 2.3. *The class of all interval orders of width at most w can be partitioned on-line into $3w-2$ chains. Furthermore, this is best possible.*

In fact, it is shown in [15] that order is not essential. It is enough to know whether elements are comparable or not, i.e., the result can be stated in terms of coloring interval graphs where only adjacencies are provided by the Builder and not an interval representation.

Theorem 2.4. *The class of all interval graphs of maximum clique size at most k can be colored on-line using $3k-2$ colors. Furthermore, this is best possible.*

Moreover, it is known that First Fit performs reasonably well in coloring interval graphs. In fact, in a series of papers [22], [11], [13], [17], [2] and [16] and [8], incremental improvements have been made in analyzing the performance of First Fit in the coloring of interval graphs with maximum clique size at most k, with the current upper bound being $8k-4$.

From below, Chrobak and Ślusarek [5] have given a computer based proof to show that when k is sufficiently large, First Fit can be forced to use more than $4.5k$ colors on an interval graph with maximum clique size k. Currently, the best lower bound is given in [14], where it is shown that for every $\epsilon > 0$, there is a k_0 so that First Fit can be forced to use $(5-\epsilon)k$ colors on an interval graph with maximum clique size k, provided $k \geq k_0$.

The fact that interval orders exclude $\mathbf{2}+\mathbf{2}$ plays a pivotal role in the following result due to Hopkins [7].

Theorem 2.5. *The on-line dimension of the class of interval orders of width at most w is at most $4w-4$.*

On the other hand, First Fit does not perform well when used as an algorithm for chain partitioning on general posets. In [10], it is shown that on a width 2 poset having $O(n^2)$ points, First Fit can be forced to use n chains. Nevertheless, Bosek, Krawczyk and Szczypka [4] showed that First Fit works surprisingly well in partitioning posets into chains provided they exclude two long incomparable chains.

Theorem 2.6. *For each $k \geq 3$, there exists a constant c_k so that the class of posets having width at most w and excluding $\mathbf{k}+\mathbf{k}$ will be partitioned into $c_k w^2$ chains using First Fit.*

We should note that this last result played a key role in Bosek and Krawczyk's proof of Theorem 2.2. However, it was noted in [4] that the inequality in Theorem 2.6 might not be tight, and quite recently, this issue has been settled by Joret and Milans [9] with the following strengthening.

Theorem 2.7. *If $r, s \geq 2$, then First Fit will use at most $8(r-1)(s-1)w$ chains in partitioning a poset into chains provided the width of the poset is at most w and it excludes $\mathbf{r}+\mathbf{s}$.*

We note that the elegant argument given by Joret and Milans is an extension of the column labeling method introduced by Pemmaraju, Raman and Varadarajan [17]. Sharpening this labeling tool was a central component in the approaches taken by Brightwell, Kierstead and Trotter in [2] and by Narayanaswary and Babu [16].

When the improved bound from Theorem 2.7 is substituted into the argument for Theorem 2.2, the new upperbound for the general on-line chain partitioning problem becomes $O(w^{14 \log w})$.

We encourage the reader to consult the recent survey paper [1] for an up-to-date discussion of results on on-line chain partitioning.

2.3. **Crowns and On-Line Dimension.** Here is a second instance where two long incomparable chains play a key role.

For integers n and k with $n \geq 3$ and $k \geq 0$, let S_n^k denote the poset of height 2 having $n+k$ maximal elements $a_1, a_2, a_3, \dots, a_{n+k}$ and $n+k$ minimal elements $b_1, b_2, b_3, \dots, b_{n+k}$. The order relation is defined (cyclically) by setting b_i to be incomparable with a_i, a_{i+1}, $a_{i+2}, \dots, a_{i+k}$ and under the remaining $n-1$ maximal elements. This family of posets are called *generalized crowns*, and Trotter [19] gave the following formula for their dimension.

Theorem 2.8. *For $n \geq 3$ and $k \geq 0$,* $\dim(S_n^k) = \lceil 2(n+k)/(k+2) \rceil$.

When $k=0$, the poset S_n^0 has $2n$ points and dimension n. It is also called the *standard example* of an n-dimensional poset, and in most settings, it is just denoted as S_n. The standard example S_n is irreducible, i.e., the removal of any point lowers the dimension of the remaining subposet to $n-1$.

When $n=3$, the posets in the family $\mathcal{F} = \{S_3^k : k \geq 0\}$ have dimension 3 and they are also irreducible. Historically, the posets in $\mathcal{F}$ were studied before this more general definition was made, and in early papers, they were called *crowns*.

In [12], the following results are proved.

Theorem 2.9. *The on-line dimension of the class of posets having width at most* 2 *is at most* 5.

Theorem 2.10. *The on-line dimension of the class of posets having width at most* 3 *is infinite.*

The proof of Theorem 2.10 provides a strategy for Builder to win the OL-Dim$(\mathcal{P}, n, t)$ game where $\mathcal{P}$ is the class of posets of width at most 3, provided n is sufficiently large in comparison to t. Builder starts by constructing two long incomparable chains, which the Assigner can force to be at least of size $t-2$. Builder then wins the game by appropriately adding elements that form $S_3^0 = S_3$, which is both a crown and a standard example.

Subsequently, Kierstead, McNulty and Trotter proved the following result.

Theorem 2.11. *The on-line dimension of the class of width at most w and excluding all crowns in $\mathcal{F} = \{S_3^k : k \geq 0\}$ has on-line dimension at most $s!$, provided the posets in the class can be partitioned on-line into s chains.*

From Theorem 2.10, we know that one must exclude the smallest crown S_3 in order to have a chance for finite on-line dimension, but it is still not known whether it is necessary to exclude all crowns when the width is allowed to be larger than 3. The proof given in [12] depends heavily on this assumption, but it may actually be the case that it is enough to exclude S_3.

Regardless, in view of Theorem 2.6 and of the proof of Theorem 2.10 the following question emerges naturally. Let k and w be positive integers with $k \geq 3$ and $w \geq 2$. Does the class of posets of width at most w and excluding $\mathbf{k}+\mathbf{k}$ have finite on-line dimension? The principal result of this paper will be the following affirmative answer.

Theorem 2.12. *Fix positive integers k and w. If the class $\mathcal{P}(w,k)$ of posets having width at most w and excluding $\mathbf{k}+\mathbf{k}$ can be partitioned on-line into s chains, then the online dimension of $\mathcal{P}(w,k)$ is at most $m!$, where $m = s(6k-11)$.*

3. Preliminaries

Here is an elementary consequence of the property that a poset excludes $\mathbf{k}+\mathbf{k}$.

Proposition 3.1. *Let $C_1 = \{x_1 < x_2 < \cdots < x_k\}$ and $C_2 = \{y_1 < y_2 < \cdots < y_k\}$ be disjoint k-element chains in a poset P excluding $\mathbf{k}+\mathbf{k}$. Then either $x_1 < y_k$ in P or $y_1 < x_k$ in P.*

In work to follow, we need a somewhat stronger version of this basic result, and we need to allow the chains to intersect. For this purpose, we make the following definition: Let P be a poset, and let $\mathbb{N}_0$ denote the set of non-negative integers. Define a function $\rho : P \times P \longrightarrow \mathbb{N}_0$ by setting (a) $\rho(x,y) = 0$ when $x \not< y$; and (b) when $x < y$ in P, $\rho(x,y)$ is the largest positive integer m for which there is a chain $x = z_1 < z_2 < \cdots < z_m < y$ in P. For emphasis, we state the following elementary property satisfied by this function.

Proposition 3.2. *If $x \leq y \leq z$ in P, then $\rho(x,z) \geq \rho(x,y) + \rho(y,z)$.*

The following elementary lemma will be key to our proof of Theorem 2.12.

Lemma 3.3. *Let $k \geq 2$, let x_1, x_2, y_1, y_2 be points in a poset P that excludes $\mathbf{k}+\mathbf{k}$. If $\rho(x_1,x_2) \geq k-1$ and $\rho(y_1,y_2) \geq k-1$, then*

$$\rho(x_1,y_2) + \rho(y_1,x_2) \geq \rho(x_1,x_2) + \rho(y_1,y_2) - 2k + 3$$

Proof. We argue by induction on the non-negative integer $q = \rho(x_1,x_2) + \rho(y_1,y_2) - 2k+2$. First consider the base case $q = 0$, where we need only show that either $x_1 < y_2$ or $y_1 < x_m$ in P. Let $n = \rho(x_1,x_2)$ and $m = \rho(y_1,y_2)$. Then choose chains $C_1 = x_1 = u_1 < u_2 < \cdots < u_n < x_2$ and $C_2 = y_1 = v_1 < v_2 < \cdots < v_m < y_2$. Without loss of generality, we may assume that $n \geq m$. If $C_1 \cap C_2 \neq \emptyset$, then $x_1 < y_2$ and $y_1 < x_2$, so we may assume that $C_1 \cap C_2 = \emptyset$. Now the conclusion that either $x_1 < y_2$ or $y_1 < x_2$ follows from the fact that P excludes $\mathbf{k}+\mathbf{k}$.

Now suppose that $q > 0$ and that the conclusion of the lemma holds for smaller values of q. We may assume without loss of generality that $n \geq m$. Since P does not contain $\mathbf{k}+\mathbf{k}$, we conclude that either $x_1 < y_2$ or $y_1 < x_2$. If $x_1 < y_2$, we apply the inductive hypothesis to the elements u_2, x_2, y_1, y_2 and conclude that $\rho(u_2, y_2) + \rho(y_1, x_2) \geq q$. Since $\rho(x_1, y_2) \geq \rho(u_2, y_2) + 1$, we conclude that

$$\rho(x_1, y_2) + \rho(y_1, x_2) \geq q + 1 = \rho(x_1, x_2) + \rho(y_1, y_2) - 2k + 3.$$

If on the other hand, $y_1 < x_2$, we apply the inductive hypothesis to x_1, u_n, y_1, y_2. Now we observe that $\rho(x_1, y_2) + \rho(y_1, u_n) \geq q$, and from this the desired inequality again follows immediately. □

4. Proof of the Main Theorem

Readers who are familiar with the proof techniques in [12] will recognize that we are adapting for our purposes the following concepts that first appear in that paper: (a) an auxiliary partial order on the chains in an on-line chain partition; (b) the construction of an on-line realizer using permutations; and (c) the notion of a blocking chain. However, there are some key moments where the proof we present will diverge from the approach of [12].

For convenience, we write $x \| y$ when x and y are distinct incomparable points of P.

Our main theorem is trivial if either $w = 1$ or $k = 1$, so we may assume $w \geq 2$ and $k \geq 2$. The case $w = 2$ is handled by Theorem 2.9, while the case $k = 2$ is handled by Theorem 2.5. So for the remainder of the proof, we fix integers $w \geq 3$ and $k \geq 3$, and we let $\mathcal{P} = \mathcal{P}(w, k)$ be the class of posets of width at most w and excluding $\mathbf{k} + \mathbf{k}$.

4.1. **Modifying the Chain Partition.** We suppose that the poset P is partitioned on-line into s chains, and denote these chains as $C_1, C_2, \ldots, C_s$. It is not important how this partition is obtained, so for example, it could be determined using the algorithm of Bosek and Krawczyk from Theorem 2.2, but it could even be given to us by a generous Builder. Regardless, we elect to modify the partition using chains of the form: $C_{i,j}$ where i and j are integers with $1 \leq i \leq s$ and $1 \leq j \leq 6k - 11$. Accordingly, there will be $s(6k - 11)$ chains in the revised scheme. Assignment to these new chains is determined by the following simple rule. When a point x enters, if the old algorithm would assign x to chain C_i, the new algorithm assigns it to chain $C_{i,j}$, using First Fit to break ties on the second coordinate so that the following key property is maintained:

The Separation Principle. If u and v are distinct points in a chain $C_{i,j}$ and $u < v$ in P, then $\rho(u, v) \geq 3k - 5$.

It is obvious that the Separation Principle can be maintained as long as we have $6k - 11$ choices for the second coordinate. This results from the fact that we need only be able to break ties with (at most) $3k - 6$ elements from C_i that are above x and (at most) $3k - 6$ elements from C_i that are below x.

4.2. **The Winning Strategy for Assigner.** Set $m = s(6k - 11)$ and $t = m!$. We show that Assigner can build an on-line realizer consisting of t linear extensions. Here is the winning strategy. First, relabel the $m = s(6k - 11)$ chains in the on-line chain partition satisfying the Separation Principle as $D_1, D_2, \ldots, D_m$. Also, when x is a point in P, we let $\phi(x)$ denote the unique subscript $\alpha \in \{1, 2, \ldots, m\}$ so that $x \in D_\alpha$. The realizer $\mathcal{R}$ will contain a linear extension L_σ for every permutation σ

of the integers in $\{1, 2, \ldots, m\}$, the set of subscript of the chains in our modified on-line chain partition. Let $\sigma = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be such a permutation. We explain how Assigner will build L_σ.

Suppose the new point x enters at round i and that during round $i-1$, the Assigner has constructed a linear extension L_σ of the poset determined by the first $i-1$ points of P. To update L_σ, we consider the set $V_\sigma(x)$ consisting of (1) all points v with $v > x$ in P together with (2) all points v with $x \| v$ in P, $v > u$ in L_σ for all u with $x > u$ in P and $\phi(x) < \phi(v)$. If $V_\sigma(x) = \emptyset$, place x at the very top of L_σ. If $V_\sigma(x) \neq \emptyset$, let v_0 be the least element of $V_\sigma(x)$ in L_σ and insert x immediately under v_0.

The following elementary property is stated for emphasis.

Proposition 4.1. *Let σ be a permutation of $\{1, 2, \ldots, m\}$. Then at every stage of the game, if u is immediately under v in L_σ, then either (a) $u < v$ in P or (b) $u \| v$ and $\phi(u) < \phi(v)$ in σ.*

The remainder of the proof consists of showing that this simple strategy produces a win for Assigner. Let x and y be incomparable points in P. We show that there is some L_σ for which $x > y$ in L_σ. By symmetry, this is enough to show that Assigner will maintain $\mathcal{R}$ as an on-line realizer. To accomplish this task, we freeze the poset at the first moment in time that both x and y are present and argue about the poset P that we have at that stage. Let $\phi(y) = a$ and $\phi(x) = b$. We will restrict our attention to those L_σ for which a is the first element in σ and b is the last. In any such L_σ, when x enters, it will go as high as possible, and when y enters, it will go as low as possible.

We consider points in the following subposets of P:

$$U = \{u : u < y \text{ in } P \text{ and } u \| x\} \quad \text{and} \quad V = \{v : v \| y, v \not\leq x\}.$$

For each $u \in U$, note that $\rho(u, y) > 0$. Also, if $\rho(u, y) = q$, and $u = u_1 < u_2 < \cdots < u_q < y$ is a chain, then $u_2, u_3, \ldots, u_q$ all come from U.

Dually, for each $v \in V$, we let $h(v)$ be the largest integer r for which there is a chain $v_1 < v_2 < v_3 < \cdots < v_r = v$ with all elements in this chain coming from $V \cup \{x\}$. Since x and all elements of V are incomparable with y, we have the following elementary observation.

Proposition 4.2. *Let $u \in U$ and $v \in V$. If $\rho(u, y) \geq k-1$ and $h(v) \geq k$, then $u < v$ in P.*

Let $S = \{1, 2, \ldots, m\} \backslash \{a, b\}$, i.e., S is the set of all subscripts of chains, excepting the chains containing x and y respectively. We will now determine an auxiliary partial order Q on S. Subsequently we will show that we must have $x > y$ in any L_σ with a as its least element, b as its greatest element, and the $m-2$ elements of S ordered by *any* linear extension of Q.

Let $S_U = \{\alpha \in S : \phi(u) = \alpha \text{ for some } u \in U\}$, and let $S_V = \{\alpha \in S : \phi(v) = \alpha$ for some $v \in V\}$. The set $S_U \cup S_V$ is then partitioned as $S_U \cup S_V = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$, according to the following scheme:

(1) $\alpha \in S_1$ if $\alpha \in S_U - S_V$.
(2) $\alpha \in S_2$ if (a) $\alpha \in S_U \cap S_V$ and (b) there is some u in U with $\phi(u) = \alpha$ and $\rho(u, y) < k-1$.
    Note that if $v \in V$ with $\phi(v) = \alpha$, then at least $2k-3$ elements of the

longest chain connecting u and v are in V. Therfore $h(v) \geq 2k-2$ for any such v.

(3) $\alpha \in S_3$ if (a) $\alpha \in S_U \cap S_V$, (b) $\rho(u,y) \geq k-1$ for any $u \in U$ with $\phi(u) = \alpha$, and (c) $h(v) \geq k$ for any $v \in V$ with $\phi(v) = \alpha$.
Note that if $\alpha, \beta \in S_3$, then $u < v$ in P for every $u \in U$ and $v \in V$ with $\phi(u) = \alpha$ and $\phi(v) = \beta$.

(4) $\alpha \in S_4$ if (a) $\alpha \in S_U \cap S_V$, (b) there is some $v \in V$ with $\phi(v) = \alpha$ and $h(v) \leq k-1$.
Note that we must have $\rho(u,y) \geq 2k-3$ for any $u \in U$ with $\phi(u) = \alpha$.

(5) $\alpha \in S_5$ if $\alpha \in S_V - S_U$.

Next, we define a binary relation Q on S by the following rules:

(1) Put (α,β) in Q if there are integers i, j with $1 \leq i < j \leq 5$ so that $\alpha \in S_i$ and $\beta \in S_j$.

(2) Let $i \in \{2,4\}$ and let $\alpha, \beta \in S_i$. Put (α,β) in Q if there are elements $u \in U$, $v \in V$ so that (a) $\phi(u) = \alpha$, (b) $\phi(v) = \beta$, and (c) $\rho(u,v) < k-1$.

Proposition 4.3. *The binary relation satisfies the following properties and is therefore a partial order on S:*

(1) *Q is irreflexive, i.e., $(\alpha,\alpha) \notin Q$, for every $\alpha \in S$.*
(2) *Q is asymmetric, i.e., if $(\alpha,\beta) \in Q$, then $(\beta,\alpha) \notin Q$.*
(3) *Q is transitive, i.e., if $(\alpha,\beta) \in Q$ and $(\beta,\gamma) \in Q$, then $(\alpha,\gamma) \in Q$.*

Proof. The first property follows directly from the definition of Q and the sparcity of the chains. Clearly, to prove the second and third properties, it is enough to show that they hold for the restriction of Q to S_i, for $i = 2$ and $i = 4$; and a single argument suffices for this purpose. Fix $i \in \{2,4\}$, and let $\alpha, \beta, \gamma \in S_i$ and suppose that both (α,β) and (β,γ) belong to Q (note that we allow $\alpha = \gamma$). Choose $u_\alpha \in U$ and $v_\beta \in V$ that witness $(\alpha,\beta) \in Q$. Also choose $u_\beta \in U$ and $v_\gamma \in V$ that witness $(\beta,\gamma) \in Q$. Then $\rho(u_\alpha, v_\beta) \leq k-2$ and $\rho(u_\beta, v_\gamma) \leq k-2$. Since $u_\beta < v_\beta$ in P, $\rho(u_\beta, v_\beta) \geq 3k-5$. If $\alpha = \gamma$, then $\rho(u_\alpha, v_\gamma) \geq 3k-5$; if $\alpha \neq \gamma$ and $(\alpha,\gamma) \notin Q$, then $\rho(u_\alpha, v_\gamma) \geq k-1$. We conclude that

$$2(k-2) \geq \rho(u_\alpha, v_\beta) + \rho(u_\beta, v_\gamma) \geq (3k-5) + (k-1) - 2k + 3 = 2k-3.$$

The contradiction completes the proof. $\square$

Now that we have shown that Q is a partial order on S, we will write $\alpha < \beta$ in Q rather than $(\alpha,\beta) \in Q$. Next, let $\sigma_0 = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{m-1}, \alpha_m)$ be any permutation of $\{1, 2, \ldots, m\}$ so that $\alpha_1 = 1 = \phi(y)$, $\alpha_m = m = \phi(x)$, and $\alpha_2 < \alpha_3 < \cdots < \alpha_{m-1}$ is a linear extension of Q. We will now proceed to show that $x > y$ in L_{σ_0}.

We start with an easy but important lemma. The reader may note that the conclusion of this lemma was precisely the motivation for our definition of the auxiliary partial order Q.

Lemma 4.4. *Let $u \in U$ and $v \in V$. If $u \| v$ in P, then $\phi(u) < \phi(v)$ in Q.*

Proof. Let $u \in U$ and $v \in V$ and set $\alpha = \phi(u)$ and $\beta = \phi(v)$. We assume that $u \| v$ in P and show that $\alpha < \beta$ in Q. Choose integers i and j so that $\alpha \in S_i$ and $\beta \in S_j$. Note that $j \neq 1$ and $i \neq 5$. Furthermore, the conclusion of the lemma holds if $i < j$ so we may assume that $i \geq j$.

Suppose first that $i = j$. Since $\rho(u, v) = 0$, the conclusion of the lemma is witnessed by u and v when $i = j = 2$ and when $i = j = 4$. However, $\rho(u, v) = 0$ implies that we cannot have $i = j = 3$ (see the comment made when S_3 was first defined). So we may assume that $i \neq j$, i.e., $i > j$. Then i is either 3 or 4 and j is either 2 or 3. From the definitions of S_2, S_3 and S_4, we conclude that $\rho(u, y) \geq k-1$ and $h(v) \geq k$. However, these statements imply that $u < v$ in P. The contradiction completes the proof. □

4.3. **Blocking Chains.** For the remainder of the proof, we assume that $x < y$ in L_{σ_0} and argue to a contradiction. We say that a sequence $x = z_0, z_1, z_2, \ldots, z_r = y$ of points is a *blocking chain* when for each $i = 0, 1, 2, \ldots, r-1$, z_i precedes z_{i+1} in L_{σ_0}, and either $z_i < z_{i+1}$ in P or $z_i \| z_{i+1}$ and $\phi(z_i) < \phi(z_{i+1})$ in σ_0. Note that the string of all elements in L_{σ_0} beginning with x and ending with y forms a blocking chain. Now consider a blocking chain $x = z_0, z_1, z_2, \ldots, z_r$ with r as small as possible.

Lemma 4.5. *Let $x = z_0, z_1, z_2, \ldots, z_r = y$ be a blocking chain with r as small as possible. Then the following statements hold:*

(1) *The integer r is odd and $\phi(z_i) \neq \phi(z_j)$ whenever $0 \leq i < j \leq r$.*
(2) *When i is even and $0 \leq i < r$, $z_i < z_{i+1}$ in P and $\phi(z_i) > \phi(z_{i+1})$ in σ_0.*
(3) *When i is odd and $1 \leq i < r$, $z_i \| z_{i+1}$ and $\phi(z_i) < \phi(z_{i+1})$ in σ_0.*
(4) *When $0 \leq i < j \leq r$ and $j \geq i+2$, $z_i \| z_j$ and $\phi(z_i) > \phi(z_j)$ in σ_0.*

Proof. This proof is the same as in the argument given in [12]. First note that $x < z_1$ and $z_{r-1} < y$ because $\phi(y)$ is first and $\phi(x)$ is last in σ_0. Since we choose a shortest blocking chain there is no i with $z_{i-1} < z_i < z_{i+1}$ nor $z_{i-1} \| z_i \| z_{i+1}$. This yields (2) and (3) and implies that r is odd. If $\phi(z_i) = \phi(z_{i+1})$ we can skip z_i. If $i+1 < j$ and either $\phi(z_i) = \phi(z_j)$ or $\phi(z_i) < \phi(z_j)$ there is a shorter blocking chain. This yields the remaining piece for (1) and (4). □

To complete the proof, we note that $z_{r-1} \in U$ while $z_{r-2} \in V$. Since $z_{r-1} \| z_{r-2}$, it follows from Lemma 4.4 that $\phi(z_{r-1}) < \phi(z_{r-2})$, which is a contradiction.

4.4. **An Alternate Approach.** As we are very much interested in extending the results and techniques developed here in new directions, we comment briefly that there is an alternative approach to proving our principal theorem. When P excludes $\mathbf{k} + \mathbf{k}$, we may define a partial order P_I so that:

(1) If $x < y$ in P_I, then $x < y$ in P, i.e., P is an extension of P_I.
(2) If $\rho(x, y) \geq (k-1)(w-1)$ in P, then $x < y$ in P_I.
(3) P_I is an interval order.

The definition of P_I can be made using the following rule: For each $x \in P$, associate with x the closed interval $[d_x, u_x]$ where $d_x = |\{y \in P : y < x \text{ in } P\}|$ and $u_x = |P| - |\{z \in P : z > x \text{ in } P\}|$. With this approach, we need to use more values $(w(k-1))$ for the second coordinate in order to establish the Separation Principle. However, the definition of the auxiliary order is simpler. Now it is enough to take $\alpha < \beta$ in Q when there exist points $u \in U$ and $v \in V$ with $\phi(u) = \alpha$, $\phi(v) = \beta$ and $u \| v$ in P_I.

5. Concluding Remarks

The techniques we have introduced in this paper may shed some light on the question first raised in [12]:

Question 5.1. *Fix an integer $w \geq 3$. Is the on-line dimension of the class of all posets of width at most w and excluding the standard example S_3 finite?*

In [12] as well as in this paper, upper bounds are established on the on-line dimension of a class of posets that have the general form $s!$ where s is the number of chains in an on-line chain partition. In [12], an exponential lower bound was produced, but we do not see how to apply those techniques in this setting, and it may indeed be the case that Assigner can actually construct an on-line realizer of much more modest size for the class of posets of width at most k and excluding $\mathbf{k} + \mathbf{k}$.

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