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THE ORDER DIMENSION OF CONVEX POLYTOPES*

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Abstract. With a convex polytope $\mathbf{M}$ in $\mathbb{R}^3$, a partially ordered set $\mathbf{P_M}$ is associated whose elements are the vertices, edges, and faces of $\mathbf{M}$ ordered by inclusion. This paper shows that the order dimension of $\mathbf{P_M}$ is exactly 4 for every convex polytope $\mathbf{M}$. In fact, the subposet of $\mathbf{P_M}$ determined by the vertices and faces is critical in the sense that deleting any element leaves a poset of dimension 3.

Key words. convex polytopes, planar graphs, dimension

AMS(MOS) subject classifications. 06A07, 05C35

1. Introduction. We consider a planar map $\mathbf{M}$ as a finite connected planar graph $\mathbf{G} = (V, E)$ together with a plane drawing D of $\mathbf{G}$, i.e., a representation of $\mathbf{G}$ by points and arcs in the plane $\mathbb{R}^2$ in which there are no edge crossings. We do not distinguish between a vertex (edge) of $\mathbf{G}$ and the corresponding point (arc) in the plane. Deleting the vertices and edges of $\mathbf{G}$ from the plane leaves several connected components whose closures are the *faces* of $\mathbf{M}$. The unique unbounded face is called the *exterior* or *outside* face.

With a planar map $\mathbf{M}$, we associate a partially ordered set (poset) $\mathbf{P_M}$ whose elements are the vertices, edges, and faces (including the exterior face) of $\mathbf{M}$ ordered by inclusion. As an example, a planar map $\mathbf{M}$ and its associated poset $\mathbf{P_M}$ are shown in Fig. 1, below.

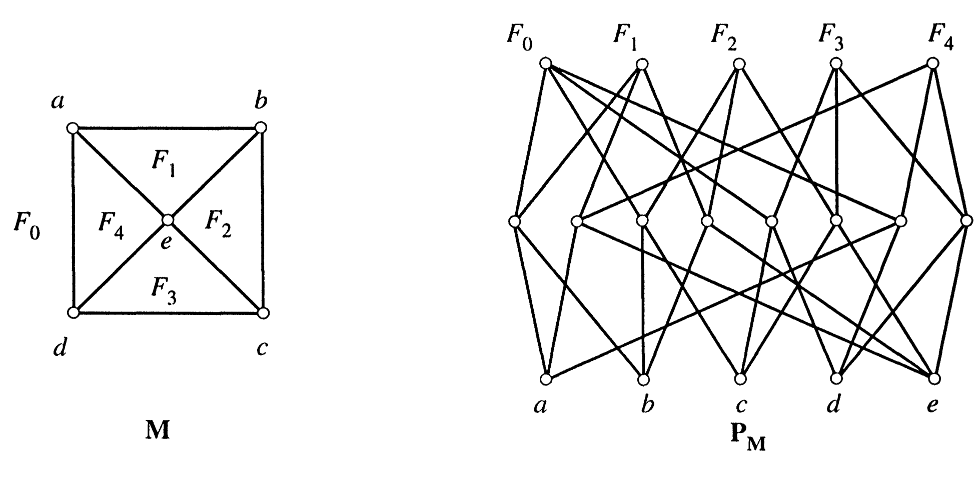

FIG. 1

With a convex polytope $\mathbf{M}$ in $\mathbb{R}^3$, there is associated a planar map, which we also denote by $\mathbf{M}$. Among all planar maps, a well-known theorem of Steinitz [13] character-

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izes those associated with convex polytopes in $\mathbb{R}^3$. These are exactly the three-connected planar maps. For example, the planar map in Fig. 1 is such a map.

Dushnik and Miller [2] defined the *order dimension* of a finite poset $\mathbf{P}$, denoted $\dim(\mathbf{P})$, as the least positive integer t for which $\mathbf{P}$ is the intersection of t linear orders. The principal result of this paper will be the following theorem.

THEOREM 1.1. *Let* $\mathbf{M}$ *be a planar map associated with a convex polytope in* $\mathbb{R}^3$, *and let* $\mathbf{P_M}$ *be the partially ordered set of vertices, edges and faces of* $\mathbf{M}$ *ordered by inclusion. Then* $\dim(\mathbf{P_M}) = 4$.

Before proceeding with the proof, we pause to make a few comments concerning the origin of this problem. Our original motivation comes from the study of convex polytopes in $\mathbb{R}^n$. The *face lattice* of a convex polytope $\mathbf{M}$ is the poset consisting of all vertices, edges, faces, hyperfaces, and so forth, partially ordered by inclusion. In Birkhoff's lattice theory book [1], the problem of determining the order dimension of the face lattice of a polytope in $\mathbb{R}^n$ is posed and is credited to Kurepa (see also Golumbic's book [3, p. 137]). In $\mathbb{R}^2$, the poset of vertices and edges of a convex polygon has the following form. The point set is $\{x_i : 1 \leq i \leq m\} \cup \{y_i : 1 \leq i \leq m\}$, and the order is given by $x_i < y_i$ and $x_i < y_{i+1}$ (cyclically) for $i = 1, 2, \ldots, m$, where $m \geq 3$ is the number of vertices. Such posets are easily seen to be three-dimensional. They belong to a well-known family of posets called *crowns* [14]. (See Fig. 2.)

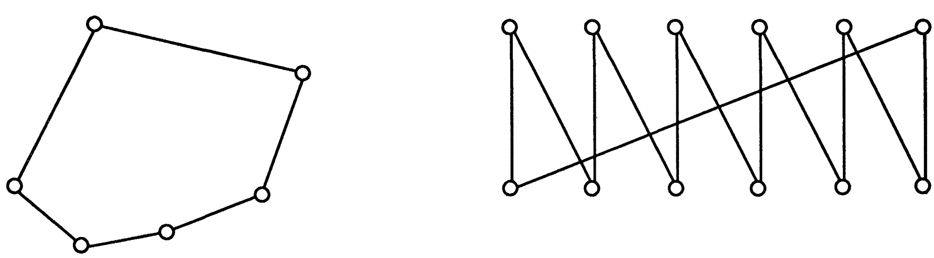

FIG. 2

If $n \geq 4$, there exist convex polytopes in $\mathbb{R}^n$ for which the face lattice has arbitrarily large dimension. This phenomenon is due to the existence of cyclical polytopes that have the property that they contain large sets of vertices each pair of which is contained in an edge. Spencer [12] showed that the order dimension $d(m)$ of the poset of all 1- and 2-element subsets of an m-element set satisfies $\log\log m \leq d(m) \leq 2\log\log m$.

Accordingly, the problem is of interest only in $\mathbb{R}^3$. Sedmak [11] reports on the existence of (nonconvex) polyhedra in $\mathbb{R}^3$ with face lattices of arbitrarily large dimension. However, our Theorem 1.1 implies that the order dimension of $\mathbf{P_M}$ is 4 whenever $\mathbf{M}$ is associated with a convex polytope in $\mathbb{R}^3$, so for example, the poset shown in Fig. 1 has order dimension 4.

Also, we are motivated by the work of Schnyder [10], who proved the following elegant characterization of planar graphs.

THEOREM 1.2. *Let* $\mathbf{G} = (V, E)$ *be a graph and let* $\mathbf{Q_G}$ *denote the poset consisting of the vertices and edges of* $\mathbf{G}$ *partially ordered by inclusion. Then* $\mathbf{G}$ *is planar if and only if the order dimension of* $\mathbf{Q_G}$ *is at most* 3.

It is relatively easy to show that $\mathbf{G}$ is planar if $\dim(\mathbf{Q_G}) \leq 3$. Schnyder's argument to show that $\dim(\mathbf{Q_G}) \leq 3$ when $\mathbf{G}$ is planar is quite complex and requires the development of some entirely new concepts for planar graphs. However, Schnyder is able to capitalize on the fact that in this part of the proof, it can be assumed that $\mathbf{G}$ is a maximal planar graph. In this case, a plane drawing of $\mathbf{G}$ without edge crossings produces a planar triangulation $\mathbf{M}$, i.e., a planar map $\mathbf{M}$ in which every face (including the exterior face) is a triangle.

It is natural to ask what happens to the order dimension of the poset associated with a planar graph if we add the faces determined by a particular drawing. It is not at all clear why the order dimension should be bounded by any absolute constant, and it is conceivable that a planar graph can be drawn as two different maps for which the associated posets have different order dimension.

In the final section of this paper, Schnyder comments that it follows easily from his Theorem 1.2 that if $\mathbf{M}$ is a convex polytope in $\mathbb{R}^3$ in which every face is a triangle, then $\dim(\mathbf{P_M}) \leq 4$. By duality, the upper bound $\dim(\mathbf{P_M}) \leq 4$ also holds if every vertex has degree 3. For these reasons, the problem of finding an upper bound (if one exists) when $\mathbf{M}$ is an arbitrary convex polytope in $\mathbb{R}^3$ is a natural one.

We comment that Schnyder's theorem can be derived easily from our results. Also, we have been successful in establishing the upper bound $\dim(\mathbf{P_M}) \leq 4$ when $\mathbf{M}$ is an arbitrary planar map—allowing loops and multiple edges. As this result requires additional machinery, it will appear in a subsequent paper. For the general theorem, the results and techniques of this paper will serve as an essential first step.

In the next section of this paper, we collect some facts from dimension theory. The major part of the proof of Theorem 1.1 in §§3 and 4 involves the construction of a family of paths in a planar map. We fix three special vertices v_1, v_2, v_3 on the outside face and then, for each other vertex x, find three vertex-disjoint paths from x to the v_i. Menger's theorem tells us that, provided no pair of vertices separates any other vertex from $\{v_1, v_2, v_3\}$, we can find such a family of paths in the graph. We show that the family we construct has certain other properties related to the plane representation of the graph. This enables us to define three partial orders on the vertex set of the map, which we use in turn to define three linear extensions of $\mathbf{P_M}$. In the fourth linear extension, we require only that the outside face is below all vertices not on that face. These four linear extensions then intersect to give $\mathbf{P_M}$.

2. Necessary tools from dimension theory. In this section, we describe briefly some basic concepts of dimension theory needed in this paper. We refer the reader to the monograph [17] by Trotter, the survey article by Kelly [5] and by Kelly and Trotter [6] and the chapters in [15], [16] by Trotter for additional background material and an extensive list of references.

Let $\mathbf{P}$ be a finite poset. We write $x\|y$ to indicate that x and y are incomparable points in $\mathbf{P}$. A family $\mathbf{F} = \{L_1, L_2, \ldots, L_t\}$ of linear extensions of $\mathbf{P}$ is called a *realizer* of $\mathbf{P}$ if $\mathbf{P} = L_1 \cap L_2 \cap \cdots \cap L_t$, i.e., $x < y$ in $\mathbf{P}$ if and only if $x < y$ in L_i for $i = 1, 2, \ldots, t$. The dimension of $\mathbf{P}$ is then the minimum cardinality of a realizer.

An ordered pair (x, y) of incomparable points is called a *critical pair* if $z < x$ implies $z < y$ and $w > y$ implies $w > x$ for all $z, w \in \mathbf{P}$. In Fig. 3, we show a critical pair in a poset.

If (x, y) is a critical pair in a poset $\mathbf{P}$ and L is a linear extension of $\mathbf{P}$, we say L *reverses* (x, y) if $y < x$ in L. A family $\{L_1, L_2, \ldots, L_t\}$ of linear extensions of $\mathbf{P}$ is a realizer of $\mathbf{P}$ if and only if, for every critical pair (x, y), there is some i so that L_i reverses (x, y).

When $\mathbf{M}$ is a planar map, y a vertex, and F a face not containing y, then (y, F) is a critical pair in $\mathbf{P_M}$. So every realizer must (at least) reverse each critical pair of this type. We let $D(\mathbf{P_M})$ denote the least positive integer for which there exist t linear extensions

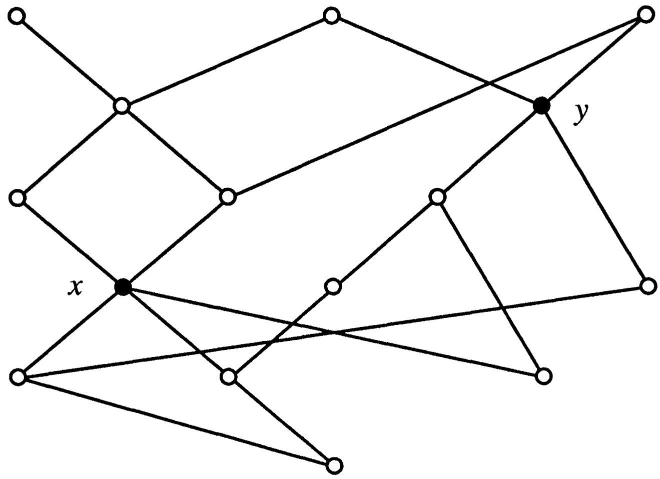

FIG. 3

$L_1, L_2, \ldots, L_t$ reversing all critical pairs of the form (y, F), where y is a vertex and F is a face not containing y. Of course, we always have $D(\mathbf{P_M}) \leq \dim(\mathbf{P_M})$.

We say that a planar map $\mathbf{M}$ is *well formed* if the critical pairs of $\mathbf{P_M}$ are exactly the pairs of the form (y, F), where y is a vertex, F is a face, and $y \notin F$. It is an easy exercise to show that if $\mathbf{M}$ is a planar map associated with a convex polytope in $\mathbb{R}^3$, then $\mathbf{M}$ is well formed so that $\dim(\mathbf{P_M}) = D(\mathbf{P_M})$.

When L is a linear order on the vertex set V of a planar map $\mathbf{M}$, y is a vertex and F is a face of $\mathbf{M}$, we write $y > F$ in L when $y > x$ in L for every vertex $x \in F$. It is easy to see that if L is any linear order on V, then there exists a linear extension L^* of $\mathbf{P_M}$ so that $y > F$ in L^* whenever $y > F$ in L. Accordingly, to show that $\dim(\mathbf{P_M}) \leq 4$ when $\mathbf{M}$ is a well-formed planar map, we must produce four linear orders, L_1, L_2, L_3, L_4 of the vertex set V so that for every critical pair (y, F), there is some i with $y > F$ in L_i.

3. Normal families of paths. When x and y are distinct vertices on the exterior face of $\mathbf{M}$, we denote by $\mathbf{M}[x, y]$ the sequence of vertices encountered in proceeding clockwise around the exterior face of $\mathbf{M}$ beginning at x and ending at y. For the sequence obtained by proceeding in a counterclockwise direction, we write $\mathbf{M}^r[x, y]$. For example, in the planar map $\mathbf{M}$ shown in Fig. 4, $\mathbf{M}[f, a] = (f, g, v_1, g, a)$ and $\mathbf{M}^r[e, a] = (e, v_2, b, a)$.

We call a triple (v_1, v_2, v_3) of distinct vertices from the exterior face of $\mathbf{M}$ a *triad* if $v_{\alpha+2} \notin M[v_\alpha, v_{\alpha+1}]$ for $\alpha = 1, 2, 3$. (Throughout this paper, subscripts are interpreted cyclically.) The triple (v_1, v_2, v_3) is a triad for the map $\mathbf{M}$, shown in Fig. 4.

When $P_1, P_2, \ldots, P_k$ are paths in $\mathbf{M}$, we denote by $S(P_1, P_2, \ldots, P_k)$ the set of all points in the plane that belong to an edge in some P_i together with those points inside any cycle formed by edges in the union of the edge sets of these paths. For example, in Fig. 4, let $P_1 = \mathbf{M}[v_1, v_2], P_2 = (c, a, g, v_1), P_3 = (c, b, d, v_2)$. Then $S(P_1, P_2, P_3)$ contains the points from the edges in these paths and points inside the triangles T_1 and T_2.

Now let $\mathbf{M}$ be a planar map and let (v_1, v_2, v_3) be a triad for $\mathbf{M}$. Let $\mathcal{F} = \{P(x, v_\alpha) : x \in V, \alpha = 1, 2, 3\}$ be a family of paths in $\mathbf{M}$. We say that $\mathcal{F}$ is a *normal family* of paths for (v_1, v_2, v_3), provided the following five properties are satisfied.

Path Property 1. For all $x \in V$ and each $\alpha = 1, 2, 3$, $P(x, v_\alpha)$ is a path from x to v_α.

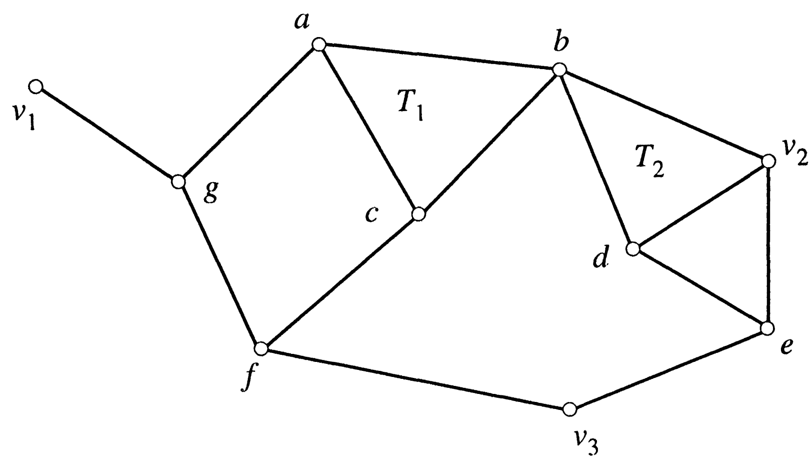

FIG. 4

Path Property 2. For all $x \in V - \{v_1, v_2, v_3\}$ and each $\alpha = 1, 2, 3$, the paths $P(x, v_\alpha)$ and $P(x, v_{\alpha+1})$ have only the vertex x in common.

Path Property 3. For each $\alpha = 1, 2, 3$, $P(v_\alpha, v_{\alpha+1}) = \mathbf{M}[v_\alpha, v_{\alpha+1}]$ and $P(v_{\alpha+1}, v_\alpha) = \mathbf{M}^r[v_{\alpha+1}, v_\alpha]$.

Path Property 4. For all $x, y \in V$ and each $\alpha = 1, 2, 3$, if $P(x, v_\alpha)$ is the path $(x = u_0, u_1, \ldots, u_t = v_\alpha)$ and $y = u_i$ for some i, then $P(y, v_\alpha)$ is the path $(y = u_i, u_{i+1}, \ldots, u_t = v_\alpha)$, i.e., $P(y, v_\alpha)$ is a terminal segment of $P(x, v_\alpha)$.

Path Property 5. For all $x \in V$ and each $\alpha = 1, 2, 3$, let $S(x, \alpha) = S(P(x, v_{\alpha+1}), P(x, v_{\alpha+2}), P(v_{\alpha+1}, v_{\alpha+2}))$. Then, for all $x, y \in V$ and each $\alpha = 1, 2, 3$, if $y \in S(x, \alpha)$, then $S(y, \alpha) \subseteq S(x, \alpha)$.

For the planar map shown in Fig. 4, it is easy to see that there are two normal families of paths for the triad (v_1, v_2, v_3). The only option is to choose $P(b, v_3)$ as either (b, c, f, v_3) or (b, d, e, v_3). We say that x and y are *α-equivalent* when $S(x, \alpha) = S(y, \alpha)$. The reader is invited to compare Schnyder's proof [10] of Theorem 1.1 and his construction of families of paths in a planar triangulation. Note that when $\mathbf{M}$ is a planar triangulation, Schnyder's argument gives an explicit construction of a normal family of paths for which there is no pair of α-equivalent vertices.

Recall that a 3-connected planar map is well formed. In the next section, we will show that a 3-connected planar map has a normal family of paths for every triad. To provide clear motivation for the concept of a normal family, we show how such a family is used to establish the upper bound $\dim(\mathbf{P_M}) \le 4$ when $\mathbf{M}$ is 3-connected. First, we will need some additional properties of normal families of paths and binary relations defined in terms of them. In what follows, let (v_1, v_2, v_3) be a triad for a planar map $\mathbf{M}$ and let $\mathcal{F} = \{P(x, v_\alpha) : x \in V, \alpha = 1, 2, 3\}$ be a normal family of paths for (v_1, v_2, v_3).

LEMMA 3.1. *If $\alpha \in \{1, 2, 3\}$, $x \in V$, $y \in S(x, \alpha)$ and $y \notin P(x, v_{\alpha+1}) \cup P(x, v_{\alpha+2})$, then $x \notin P(y, v_{\alpha+1}) \cup P(y, v_{\alpha+2})$.*

Proof. If $x \in P(y, v_{\alpha+1}) \cup P(y, v_{\alpha+2})$, then $x \in S(y, \alpha)$, so $S(x, \alpha) \subseteq S(y, \alpha)$. However, $y \in S(x, \alpha)$ and $y \notin P(x, v_{\alpha+1}) \cup P(x, v_{\alpha+2})$ require $S(y, \alpha) \subsetneqq S(x, \alpha)$. The contradiction completes the proof. $\square$

For each $\alpha \in \{1, 2, 3\}$, the binary relation Q_α defined on the vertex set V of M by $Q_\alpha = \{(x, y) : S(x, \alpha) \subsetneqq S(y, \alpha)\}$ is obviously a partial order. Note that when $S(x, \alpha) \not\subseteq S(y, \alpha)$ and $S(y, \alpha) \not\subseteq S(x, \alpha)$, we have $x \| y$ in Q_α. We simplify this by writing

$S(x,\alpha)\|S(y,\alpha)$. However, we also have $x\|y$ in Q_α when x and y are distinct α-equivalent points, i.e., $S(x,\alpha) = S(y,\alpha)$. Note that when $x\|y$ in Q_α, there is a unique $\beta \in \{\alpha+1, \alpha+2\}$, so that $S(x,\beta) \subsetneqq S(y,\beta)$.

The general plan is to take a linear extension L_α of the partial order Q_α for each $\alpha = 1,2,3$. However, we need for each L_α to satisfy certain other conditions. Ideally, we would like $x > F$ in L_α whenever $x \notin F$ and $F \subseteq S(x,\alpha)$. Since L_α extends Q_α, this will certainly occur unless F contains a vertex y, which is α-equivalent to x. Indeed, it may well be that x and y are α-equivalent, say with $y \in P(x,\alpha+1)$, and there are faces F and G in $S(x,\alpha)$ with F containing y but not x and G containing x but not y. In this situation, we clearly cannot have both $x > F$ and $y > G$ in L_α. Can we put $x > F$ in one of the other linear extensions? Not in $L_{\alpha+1}$, since $(x,y) \in Q_{\alpha+1}$. If F contains a vertex w with $(x,w) \in Q_{\alpha+2}$, then we cannot put $x > F$ in $L_{\alpha+2}$ either. Fortunately, if F contains such a vertex w, then G cannot contain a vertex z with $(y,z) \in Q_{\alpha+1}$, (see Lemma 3.4), so we may put $(y,x) \in L_\alpha$, and force $x > F$ in L_α. On the other hand, if F contains no such vertex w, then we want to put $(u,x) \in L_{\alpha+2}$ for every $u \in F$, to get $x > F$ in $L_{\alpha+2}$. We then must check that these relations do not conflict and that we can find linear extensions L_1, L_2, L_3 satisfying these various requirements.

More formally, we proceed by defining for each $\alpha \in \{1,2,3\}$ a suitable extension Q'_α of the order Q_α, and then taking a linear extension L_α of Q'_α. To accomplish this, we must first define some new binary relations on V. For each $\alpha \in \{1,2,3\}$, define $\mathcal{L}_\alpha = \{(x,y) \in V \times V : x\|y \text{ in } Q_\alpha \text{ and } S(y,\alpha+2) \subsetneqq S(x,\alpha+2)\}$ and $\mathcal{R}_\alpha = \{(x,y) \in V \times V : x\|y \text{ in } Q_\alpha \text{ and } S(y,\alpha+1) \subsetneqq S(x,\alpha+1)\}$.

Recall that the *dual* of a binary relation Q on a set V is the relation $\{(x,y) \in V \times V : (y,x) \in Q\}$. The following result is then immediate.

LEMMA 3.2. *For each $\alpha \in \{1,2,3\}$, $\mathcal{L}_\alpha$ and $\mathcal{R}_\alpha$ are partial orders on V, and $\mathcal{R}_\alpha$ is the dual of $\mathcal{L}_\alpha$.*

We think of $\mathcal{L}_\alpha$ and $\mathcal{R}_\alpha$ as denoting "left" and "right," respectively. In what follows, we will define binary relations $\mathcal{L}''_\alpha \subseteq \mathcal{L}_\alpha$ and $\mathcal{R}''_\alpha \subseteq \mathcal{R}_\alpha$; however, $\mathcal{L}''_\alpha$ and $\mathcal{R}''_\alpha$ will not be dual. First set $\mathcal{L}'_\alpha = \{(x,y) \in \mathcal{L}_\alpha$: there is a face F and a vertex $u \neq y$ such that (1) $u,x \in F$, (2) $y \in S(u,\alpha+1)$, and (3) $u \notin P(y,v_\alpha)\}$. If F and u are as above, we say that (F,u) *witnesses* $(x,y) \in \mathcal{L}'_\alpha$.

Now we set $\mathcal{L}''_\alpha = \{(x,z) \in \mathcal{L}_\alpha :$ there is some y with $(x,y) \in \mathcal{L}'_\alpha$ and $(y,z) \in Q_\alpha$ or $y = z\}$. If y is as above and (F,u) witnesses $(x,y) \in \mathcal{L}'_\alpha$, we say that the triple (F,u,y) *witnesses* $(x,z) \in \mathcal{L}''_\alpha$. Thus, $\mathcal{L}'_\alpha$ is designed to capture both of the cases discussed above, where we must impose $(x,y) \in L_\alpha$, although $(x,y) \notin Q_\alpha$, at least where $(x,y) \in \mathcal{L}_\alpha$.

We define $\mathcal{R}'_\alpha$ and $\mathcal{R}''_\alpha$ in the corresponding way. Thus we set $\mathcal{R}'_\alpha = \{(x,y) \in \mathcal{R}_\alpha$: there is a face F and a vertex $u \neq y$, such that (1) $u,x \in F$, (2) $y \in S(u,\alpha+2)$, and (3) $u \notin P(y,v_\alpha)\}$. As before, in this situation we say that (F,u) witnesses $(x,y) \in \mathcal{R}'_\alpha$. Again, just as before, we set $\mathcal{R}''_\alpha = \{(x,z) \in \mathcal{R}_\alpha :$ there is some y with $(x,y) \in \mathcal{R}'_\alpha$ and $(y,z) \in Q_\alpha$ or $y = z\}$. If here (F,u) witnesses $(x,y) \in \mathcal{R}'_\alpha$, then we say (F,u,y) witnesses $(x,z) \in \mathcal{R}''_\alpha$.

The next lemma provides some information about the binary relations $\mathcal{L}'_\alpha$ and $\mathcal{L}''_\alpha$. There is, of course, a symmetric version for $\mathcal{R}'_\alpha$ and $\mathcal{R}''_\alpha$.

LEMMA 3.3. *Let $\alpha \in \{1,2,3\}$ and suppose that (F,u,y) witnesses $(x,z) \in \mathcal{L}''_\alpha$.*

(1) *If x and z are α-equivalent, then $F \subseteq S(x,\alpha)$, and $y = z$ (i.e., $(x,z) \in \mathcal{L}'_\alpha$).*

(2) *If $S(x,\alpha)\|S(z,\alpha)$, then u and y are $(\alpha+1)$-equivalent, and both y and u are on $P(z,v_{\alpha+2})$.*

Proof. We first verify statement (1). Suppose, then, that x and z are α-equivalent. If $z \neq y$, then $S(y,\alpha) \subsetneqq S(z,\alpha) = S(x,\alpha)$, which is not possible. Thus, in this case,

$z = y$. If $F \not\subseteq S(x,\alpha)$, then $F \subseteq S(y,\alpha+1)$, so in particular $u \in S(y,\alpha+1)$. Since also $y \in S(u,\alpha+1)$, this implies that u and y are $(\alpha+1)$-equivalent, so we must have $u \in P(y,v_\alpha)$, a contradiction. This completes the proof of (1).

We now prove (2). Since $S(x,\alpha)\|S(y,\alpha)$ and $(x,y) \in \mathcal{L}_\alpha$, it is clear that $F \subseteq S(y,\alpha+1)$. Thus $S(u,\alpha+1) \subseteq S(y,\alpha+1)$. However, we also have $S(y,\alpha+1) \subseteq S(u,\alpha+1)$, so u and y are $(\alpha+1)$-equivalent. We do not have $u \in P(y,v_\alpha)$, so we must have $y \in P(u,v_\alpha)$. If $y = z$, this completes the proof, so suppose that $(y,z) \in Q_\alpha$. Then $u \in S(z,\alpha)$, but x is not in this region, so u is on $P(z,v_{\alpha+2})$. Finally, $y \in S(u,\alpha+1) \subseteq S(z,\alpha+1)$, and, since also $y \in S(z,\alpha)$, this implies that y is on $P(z,v_{\alpha+2})$. □

Note that when (F,u) witnesses $(x,y) \in \mathcal{L}'_\alpha$, the face F can be located in $S(x,\alpha)$ or in $S(x,\alpha+2)$. See Figs. 5(a) and 5(b).

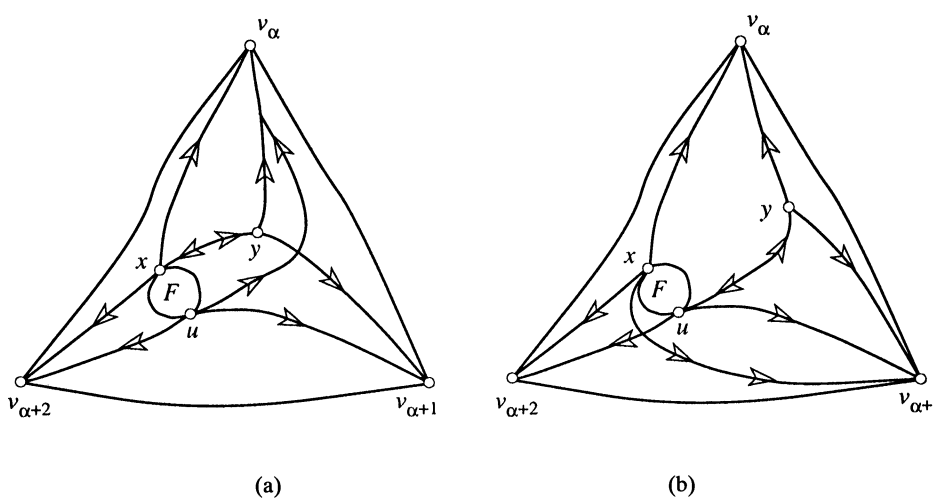

FIG. 5

Our goal is to prove that the binary relation given by $Q'_\alpha = Q_\alpha \cup \mathcal{L}''_\alpha \cup \mathcal{R}''_\alpha$ is acyclic, and then to take L'_α to be a linear extension of the transitive closure of Q'_α.

LEMMA 3.4. *If* $\alpha \in \{1,2,3\}$, $(x,z) \in \mathcal{L}''_\alpha$ *and* $(z,w) \in \mathcal{R}'_\alpha$, *then* $x \neq w$ *and* $(x,w) \in Q_\alpha \cup \mathcal{L}''_\alpha$.

Proof. Take (F,u,y), witnessing $(x,z) \in \mathcal{L}''_\alpha$, and (G,v), witnessing $(z,w) \in \mathcal{R}'_\alpha$.

First, we consider the case where $S(x,\alpha)\|S(z,\alpha)$. Let R be the region bounded by $P(x,v_\alpha)$, $P(z,v_\alpha)$, $P(z,v_{\alpha+2})$, and the clockwise path from x to u round F. Note that there are two slightly different situations, depending on whether F is in $S(x,\alpha)$ or $S(x,\alpha+2)$. (See Figs. 6(a) and 6(b).) We claim that $w \in R$.

Since z is in the interior of $S(x,\alpha+2)$ and shares a face with v, v is also in $S(x,\alpha+2)$, and hence, so is w. Also $w \in S(z,\alpha+1)$. If $F \subseteq S(x,\alpha)$, this suffices to prove our claim, so suppose that $F \subseteq S(x,\alpha+2)$. Now if $v \in S(u,\alpha+2)$, then so is w, and we are done. However, $z \notin P(u,v_{\alpha+1})$, so the only other possibility is that $v \in R$, in which case w is also in R, as required. Note that this also rules out the case where $w = x$ and $F \subseteq S(x,\alpha+2)$, since that requires $v \notin R$.

Consider the path $P = P(w,v_{\alpha+2})$ and the point it leaves R. If P joins the path $P(z,v_{\alpha+2})$ and exits via u, then $u \in S(w,\alpha)$, so $y \in S(w,\alpha)$, and hence either $(y,w) \in Q_\alpha$, when $(x,w) \in \mathcal{L}''_\alpha$, or y and w are α-equivalent when $(w,z) \in Q_\alpha$, a contradiction.

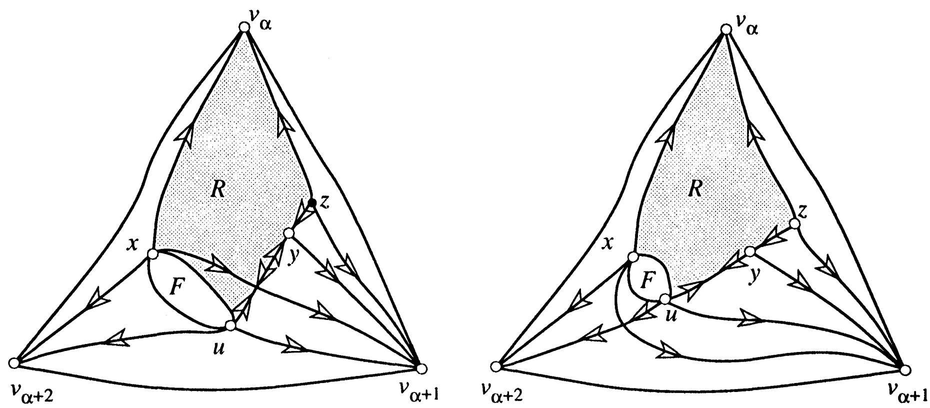

FIG. 6

The path P does not cross $P(z, v_\alpha)$, so the only remaining possibility is that it crosses $P(x, v_\alpha)$. In this case, $x \in S(w, \alpha)$, and so $(x, w) \in Q_\alpha$, unless $S(x, \alpha) = S(w, \alpha)$.

By an earlier remark, we cannot have $x = w$ and $F \subseteq S(x, \alpha + 2)$, so if $S(x, \alpha) = S(w, \alpha)$, we have $F \subseteq S(x, \alpha)$. Now v is on $P(x, v_{\alpha+1})$, but is not in $S(z, \alpha)$, since, then, $P(v, v_\alpha)$ cannot go via w. Finally, $P(v, v_{\alpha+2})$ exits R via u, but this contradicts Lemma 3.1. This completes the proof in the case where $S(x, \alpha) \| S(z, \alpha)$.

Now suppose that x and z are α-equivalent. We know that in this case $y = z$. Suppose next that z and w are also α-equivalent. If w is on $P(x, v_{\alpha+1})$, with $w \neq x$, then (F, u) witnesses $(x, w) \in \mathcal{L}'_\alpha$, so we may suppose that $w \in P(x, v_{\alpha+2})$. Then (G, v) witnesses also $(z, x) \in \mathcal{R}'_\alpha$. If $v \in S(u, \alpha + 2)$, then $x \in S(v, \alpha + 2) \subseteq S(u, \alpha + 2)$, which is clearly not possible. By symmetry, we are also done if $u \in S(v, \alpha + 1)$. So suppose $v \in S(u, \alpha + 1)$ and $u \in S(v, \alpha + 2)$. (Clearly, we cannot have, for instance, v in the interior of $S(u, \alpha)$.) Then v is in the region R bounded by $P(y, v_{\alpha+2})$, $P(u, v_\alpha)$, and F. Now consider $P(v, v_{\alpha+2})$. It cannot cross $P(u, v_\alpha)$, since that would imply $u \in S(v, \alpha + 1)$. Thus the path must join $P(y, v_{\alpha+2})$ and leave R via x. This clearly contradicts $x \in S(v, \alpha + 2)$.

Finally, suppose that x and z are α-equivalent, but that z and w are not. If $(x, w) \notin Q_\alpha$, then $(x, w) \in \mathcal{R}_\alpha$, and $x \in S(w, \alpha + 2) = S(v, \alpha + 2)$. If v is on $P(x, v_\alpha)$, then so is w, which implies $(x, w) \in Q_\alpha$. If v is not on $P(x, v_\alpha)$, then (G, v) witnesses $(z, x) \in \mathcal{R}'_\alpha$, which we have just seen is not possible. □

Now for each $\alpha = 1, 2, 3$, let $Q'_\alpha = Q_\alpha \cup \mathcal{L}''_\alpha \cup \mathcal{R}''_\alpha$. We will show that Q'_α is an acyclic binary relation on V so that the transitive closure of Q'_α is a partial order extending Q_α.

LEMMA 3.5. *For each $\alpha = 1, 2, 3$, the binary relation Q'_α is acyclic.*

Proof. Suppose to the contrary that Q'_α is not acyclic and choose a sequence $x_1, x_2, \ldots x_s$ so that $(x_i, x_{i+1}) \in Q'_\alpha$ for $i = 1, 2, \ldots, s$. Without loss of generality, we may assume that this sequence has been chosen so that s is minimum. Then the points $x_1, x_2, \ldots, x_s$ are all distinct. Furthermore, $(x_i, x_{i+2}) \notin Q'_\alpha$ for $i = 1, 2, \ldots, s$.

Since Q_α is acyclic, we know that at least one of the pairs in $\{(x_i, x_{i+1}) : 1 \leq i \leq s\}$ belongs to $\mathcal{L}''_\alpha \cup \mathcal{R}''_\alpha$. By symmetry, we will assume one (or more) of these pairs is in $\mathcal{L}''_\alpha$.

Since $\mathcal{L}''_\alpha \subseteq \mathcal{L}_\alpha$, we know that the relation $\mathcal{L}''_\alpha$ is acyclic. It follows that there is some $i \leq s$ for which $(x_i, x_{i+1}) \in \mathcal{L}''_\alpha$ and $(x_{i+1}, x_{i+2}) \in Q_\alpha \cup \mathcal{R}''_\alpha$. If $(x_{i+1}, x_{i+2}) \in Q_\alpha$, then (x_i, x_{i+2}) is clearly in $\mathcal{L}''_\alpha$; whereas if (G, v, y) witnesses $(x_{i+1}, x_{i+2}) \in \mathcal{R}''_\alpha$, then by the previous lemma we have $(x_i, y) \in Q_\alpha \cup \mathcal{L}''_\alpha$, so $(x_i, x_{i+2}) \in Q_\alpha \cup \mathcal{L}''_\alpha$. □

With the preceding lemma, we are now ready to establish the upper bound, $\dim(\mathbf{P_M}) \leq 4$, when $\mathbf{M}$ is a 3-connected planar map—under the assumption that $\mathbf{M}$ has a normal family of paths.

THEOREM 3.6. *Let (v_1, v_2, v_3) be a triad for a planar map $\mathbf{M}$ and suppose that $\mathcal{F} = \{P(x, v_\alpha) : x \in V, \alpha = 1, 2, 3\}$ is a normal family of paths for (v_1, v_2, v_3); then $D(\mathbf{P_M}) \leq 4$.*

Proof. As before, for each $\alpha = 1, 2, 3$, let Q'_α be the acyclic binary relation on the vertex set V defined by $Q'_\alpha = Q_\alpha \cup \mathcal{L}'_\alpha \cup \mathcal{R}'_\alpha$. Then the transitive closure of Q'_α is a partial order on V. Let L_α be a linear extension of this partial order. Then let L_4 be any linear order on V for which $x < y$ in L_4 whenever x is on the exterior face of $\mathbf{M}$ and y is not.

Now let (y, F) be a critical pair in $\mathbf{P_M}$. We show that $y > F$ in some L_i. If F is the exterior face, then $y > F$ in L_4. So we assume F is an interior face. In this case, we actually prove a stronger statement. We show that there is some $\alpha \in \{1, 2, 3\}$ for which $(x, y) \in Q'_\alpha$ for every $x \in F$. For such an α, we have $y > F$ in L_α.

To see this, choose $\alpha \in \{1, 2, 3\}$ so that $F \subseteq S(y, \alpha)$. Then $S(x, \alpha) \subseteq S(y, \alpha)$ for every $x \in F$. If $S(x, \alpha) \subsetneq S(y, \alpha)$ for every $x \in F$, then $y > F$ in Q_α, and thus $y > F$ in Q'_α. So we may assume that there is a point $x_0 \in F$ for which $S(x_0, \alpha) = S(y, \alpha)$. By symmetry, we may assume that $(x_0, y) \in \mathcal{L}_\alpha$. If F contains a point u for which $S(y, \alpha + 1) \subseteq S(u, \alpha + 1)$, then (F, u) witnesses $(x, y) \in \mathcal{L}'_\alpha$ for every $x \in F$ with $x \| y$ in Q_α. For any other $x \in F$, we have $S(x, \alpha) \subsetneq S(y, \alpha)$ and $(x, y) \in Q_\alpha$. Together, these statements imply $(x, y) \in Q'_\alpha$ for every $x \in F$.

It remains only to consider the case where F contains no point u for which $S(y, \alpha + 1) \subseteq S(u, \alpha + 1)$. In this case, we claim that $y > F$ in $Q'_{\alpha+1}$. To see this, observe that for each $x \in F$, either $S(x, \alpha + 1) \subsetneq S(y, \alpha + 1)$ or $x \| y$ in $Q_{\alpha+1}$. However, when $x \| y$ in $Q_{\alpha+1}$, the face F and the vertex x_0 witness $(x, y) \in \mathcal{R}'_{\alpha+1}$. Then $(x, y) \in Q'_{\alpha+1}$ for every $x \in F$. □

Since $D(\mathbf{P_M}) = \dim(\mathbf{P_M})$ when $\mathbf{M}$ is 3-connected, Theorem 3.6 yields the upper bound of our principal theorem once we have established the existence of a normal family of paths.

4. Constructing normal families of paths. Let $\mathbf{M}$ be a planar map, and let X, Y, and Z be vertices or sets of vertices in $\mathbf{M}$, with $X \cap Z = \emptyset$. We say that Z *separates* X from Y if every path in $\mathbf{M}$ from X to Y includes a vertex in Z.

Let $\mathbf{M}$ be a planar map and let (v_1, v_2, v_3) be a triad for $\mathbf{M}$. We say that $\mathbf{M}$ satisfies the *star-property* for (v_1, v_2, v_3) if for every vertex $x \in V - \{v_1, v_2, v_3\}$, no pair $\{y, z\} \subseteq V - \{x\}$ separates x from $\{v_1, v_2, v_3\}$. From Menger's theorem, it follows that $\mathbf{M}$ satisfies the star-property for (v_1, v_2, v_3) if and only if there is a family $\{P(x, v_\alpha) : x \in V, \alpha = 1, 2, 3\}$ satisfying Path Properties 1 and 2.

LEMMA 4.1 (normal family lemma). *Let $\mathbf{M}$ be a planar map and let (v_1, v_2, v_3) be a triad for $\mathbf{M}$. Then $\mathbf{M}$ has a normal family of paths for (v_1, v_2, v_3) if and only if $\mathbf{M}$ satisfies the star-property for (v_1, v_2, v_3).*

Proof. As noted previously, necessity follows from consideration of Path Properties 1 and 2 alone. We now prove sufficiency. We proceed by induction on the sum $\mathbf{S(M)}$ of the number of edges and the number of faces of $\mathbf{M}$. The lemma is true for the two maps (K_3 and $K_{1,3}$) where $\mathbf{S(M)}$ is at most 5.

So we consider a planar map $\mathbf{M}$, having $\mathbf{S}(\mathbf{M}) > 5$, with a triad (v_1, v_2, v_3) and we assume that the lemma holds for all planar maps $\mathbf{M}'$ with $\mathbf{S}(\mathbf{M}') < \mathbf{S}(\mathbf{M})$.

The remainder of the argument is organized into a series of cases. In treating these cases, we will consider maps $\mathbf{M}_0, \mathbf{M}_1, \mathbf{M}_2$, and so forth. These maps are either submaps of $\mathbf{M}$ or are formed by making minor changes in submaps of $\mathbf{M}$. When working with such a map, say $\mathbf{M}_i$, we will use the notation $\mathcal{F}_i$ for a normal family in $\mathbf{M}_i$, and a path from x to y in $\mathbf{M}_i$ will be denoted $P_i(x, y)$. The vertex set of $\mathbf{M}_i$ will be denoted V_i, and so forth. If $P(x, y)$ and $P(y, z)$ are paths having only the vertex y in common, we denote by $P(x, y) \oplus P(y, z)$ the path from x to z formed by their concatenation. We also use the notation $P(x, y) \oplus P(z, w)$ for the path formed by the union of two vertex disjoint paths for which yz is an edge.

Case 1. $\mathbf{M}$ has a cut-vertex.

Suppose $\mathbf{M} - \{x\}$ is the union of r components $C_1, C_2, \ldots, C_r$ with $r \geq 2$. If C_i is one of these components and $C_i \cap \{v_1, v_2, v_3\} = \emptyset$, then any vertex in C_i is separated from $\{v_1, v_2, v_3\}$ by x. So each C_i contains at least one element from $\{v_1, v_2, v_3\}$. Since $r \geq 2$, we may assume without loss of generality that C_1 contains exactly one element from $\{v_1, v_2, v_3\}$, say v_α. If v_α is not the only element of C_1, choose a point $y \in C_1 - \{v_\alpha\}$. Then y is separated from $\{v_1, v_2, v_3\}$ by x and v_α. So it follows that v_α is the only element of C_1 and that the edge $e = v_\alpha x$ is a bridge.

Clearly, $\mathbf{M}_0 = \mathbf{M} - \{v_\alpha\}$ satisfies the star-property for the triad $(x, v_{\alpha+1}, v_{\alpha+2})$. Now let $\mathcal{F}_0$ be a normal family of paths in $\mathbf{M}_0$. Then define $\mathcal{F}$ by $P(y, v_\alpha) = P_0(y, x) \oplus (x, v_\alpha)$ for every $y \in V - \{v_\alpha\}$, while $P(v_\alpha, v_\alpha)$ is trivial.

It is straightforward to verify that $\mathcal{F}$ is a normal family for $\mathbf{M}$. The only difficulty is to make sure that $P(x, v_{\alpha+1})$ and $P(x, v_{\alpha+2})$ have no vertex in common other than x. However, if z is common to these paths, then x is separated from $\{v_1, v_2, v_3\}$ by v_α and z. So in the remainder of the proof, we will assume $\mathbf{M}$ has no cut-vertices.

Case 2. For some $\alpha \in \{1, 2, 3\}, v_\alpha v_{\alpha+1}$ is an edge in $\mathbf{M}$.

Consider the planar map $\mathbf{M}_0$ obtained by deleting the edge $v_\alpha v_{\alpha+1}$ from $\mathbf{M}$. It is easy to show that (v_1, v_2, v_3) is a triad for $\mathbf{M}_0$ and $\mathbf{M}_0$ satisfies the star-property for (v_1, v_2, v_3). Let $\mathcal{F}_0$ be a normal family of paths for (v_1, v_2, v_3) in $\mathbf{M}_0$. Construct $\mathcal{F}$ from $\mathcal{F}_0$ by setting $P(v_\alpha, v_{\alpha+1}) = (v_\alpha, v_{\alpha+1})$ and $P(v_{\alpha+1}, v_\alpha) = (v_{\alpha+1}, v_\alpha)$ as required by Path Property 3. All other paths are the same in $\mathcal{F}$ as in $\mathcal{F}_0$. Clearly, $\mathcal{F}$ is a normal family of paths for (v_1, v_2, v_3), so in what follows, we assume that $\{v_1, v_2, v_3\}$ is an independent set.

Now we pause to make an important observation about the faces of $\mathbf{M}$. If F is an interior face, then the boundary of F is a simple cycle. If we label the vertices of F as $x_1, x_2, \ldots, x_t$ in clockwise order, then $x_i x_{i+1}$ is an edge for each i, but these are the only edges among the vertices of F. For if $x_i x_j$ is an edge, and these vertices are not consecutive, then one of x_{i+1} and x_{j+1} is an interior vertex separated from $\{v_1, v_2, v_3\}$ by x_i and x_j.

Also, a similar argument shows that if F and G are interior faces having one or more common vertices, then their common vertices occur consecutively on their boundaries.

Case 3. For some $\alpha \in \{1, 2, 3\}$, there exists an interior face F that contains v_α and a point from $\mathbf{M}[v_{\alpha+1}, v_{\alpha+2}]$.

Label the points on the boundary of F in clockwise order $x_1, x_2, \ldots, x_t$ so that x_1 belongs to $\mathbf{M}[v_{\alpha+1}, v_{\alpha+2}]$ but x_t does not. Let i be the largest integer for which $x_i \in \mathbf{M}[v_{\alpha+1}, v_{\alpha+2}]$. Then either $i = 1$ or $i = 2$, for if $i \geq 2$, then x_2 is separated from $\{v_1, v_2, v_3\}$ by x_1 and x_3.

Suppose next that $x_1 = v_{\alpha+1}$. Choose a vertex $x \in \mathbf{M}[v_\alpha, v_{\alpha+1}]$ with $x \notin \{v_\alpha, v_{\alpha+1}\}$. Then x is separated from $\{v_1, v_2, v_3\}$ by v_α and $v_{\alpha+1}$. The contradiction shows $x_1 \neq v_{\alpha+1}$. Similarly, $x_i \neq v_{\alpha+2}$.

The removal of x_i and v_α from $\mathbf{M}$ disconnects the map and leaves $v_{\alpha+1}$ in a component C_1. We let $\mathbf{M}_1$ be the submap generated by the vertices in C_1 together with x_i and v_α. Then $(v_\alpha, v_{\alpha+1}, x_i)$ is a triad for $\mathbf{M}_1$, and $\mathbf{M}_1$ satisfies the star-property for $(v_\alpha, v_{\alpha+1}, x_i)$.

The map $\mathbf{M}_2$ is formed in an analogous fashion considering the component C_2 containing $v_{\alpha+2}$ when x_1 and v_α are removed. Then $\mathbf{M}_2$ satisfies the star-property for the triad $(v_\alpha, x_1, v_{\alpha+2})$.

Now let $\mathcal{F}_1$ be a normal family in $\mathbf{M}_1$ for $(v_\alpha, v_{\alpha+1}, x_i)$, and let $\mathcal{F}_2$ be a normal family in $\mathbf{M}_2$ for $(v_\alpha, x_1, v_{\alpha+2})$. Define the normal family $\mathcal{F}$ in $\mathbf{M}$ as follows. For a vertex $x \in C_1$ with $x \neq v_\alpha$, set $P(x, v_\alpha) = P_1(x, v_\alpha)$ and $P(x, v_{\alpha+1}) = P_1(x, v_{\alpha+1})$ while $P(x, v_{\alpha+2}) = P_1(x, x_i) \oplus \mathbf{M}[x_i, v_{\alpha+2}]$. For a vertex $y \in C_2$ with $x \neq v_\alpha$, $P(y, v_\alpha) = P_2(y, v_\alpha)$ and $P(y, v_{\alpha+2}) = P_2(y, v_{\alpha+2})$ while $P(y, v_{\alpha+1}) = P_2(y, x_1) \oplus \mathbf{M}^r(x_1, v_{\alpha+1})$. If $i = 1$, we may choose $P(x_1, v_\alpha)$ as either $\mathbf{M}_1[x_1, v_\alpha]$ or $\mathbf{M}_2^r[x_1, v_\alpha]$.

It is straightforward to verify that $\mathcal{F}$ is a normal family for (v_1, v_2, v_3), so in the remainder of the proof we will assume that there is no interior face containing some v_α and a vertex from $\mathbf{M}[v_{\alpha+1}, v_{\alpha+2}]$.

A set $\{F_1, F_2, F_3\}$ of three distinct faces is called a *ring* if there exists a simple cycle C with the following three properties:

1. Every edge of C belongs to exactly one of the faces F_1, F_2, F_3.
2. No point in the interior of C belongs to the interior of any of the three faces F_1, F_2, F_3.
3. If $\alpha \in \{1, 2, 3\}$ and v_α is a vertex on C, then there is some $i \in \{1, 2, 3\}$ for which $v_\alpha \in F_i \cap F_{i+1}$.

Note that in the definition of a ring, we allow one of the three faces to be the exterior face. Also note that the cycle C is uniquely determined.

Case 4. $\mathbf{M}$ has a ring $\{F_1, F_2, F_3\}$.

Let C be the uniquely determined cycle that demonstrates that $\{F_1, F_2, F_3\}$ is a ring. Then there exist unique vertices u_1, u_2, u_3 on C so that u_i belongs to F_i and F_{i+1} for $i = 1, 2, 3$.

For each $i = 1, 2, 3$, let $u_i' = u_i$ if u_i has two or more neighbors outside C, i.e., u_i is the unique point shared by F_i and F_{i+1} in $\mathbf{M}$. Otherwise, let u_i' be the unique neighbor of u_i outside C. In this situation, u_i' also belongs to F_i and F_{i+1}.

We illustrate these definitions in Fig. 7. For the map shown, $\{F_1, F_2, F_3\}$ is a ring and the cycle $C = \{u_1, a, u_2, u_3, c\}$.

Let $\mathbf{M}_0$ be the submap of $\mathbf{M}$ induced by the vertices inside and on the cycle C. We may assume that the faces F_1, F_2, and F_3 have been labeled so that (u_1, u_2, u_3) is a triad for $\mathbf{M}_0$, i.e., $u_{\alpha+2} \notin \mathbf{M}_0[u_\alpha, u_{\alpha+1}]$ for each $\alpha = 1, 2, 3$. We now observe that $\mathbf{M}_0$ satisfies the star-property for (u_1, u_2, u_3). To see that this statement is valid, let $x \in V_0 - \{u_1, u_2, u_3\}$. In the map $\mathbf{M}$, there are three paths P_1, P_2, P_3 so that P_α is a path from x to v_α and $P_\alpha \cap P_{\alpha+1} = \{x\}$ for each $\alpha = 1, 2, 3$. It is clear that there is some β for which $u_\alpha \in P_{\alpha+\beta}$ for each $\alpha = 1, 2, 3$. Thus the initial segments of P_1, P_2, and P_3 show that $\mathbf{M}_0$ satisfies the star-property for the triad (u_1, u_2, u_3). By the inductive hypothesis, there is a normal family of paths $\mathcal{F}_0$ in $\mathbf{M}_0$ for (u_1, u_2, u_3).

Next, let $\mathbf{M}_1$ be the submap of $\mathbf{M}$ induced by the vertices outside C together with those elements of $\{u_1', u_2', u_3'\}$ that are on C. Then form $\mathbf{M}_2$ from $\mathbf{M}_1$, by adding a new vertex u_0 in the area formerly occupied by the interior of C and making u_0 adjacent to

u_1', u_2' and u_3'. The modified faces adjacent to u_0 in $\mathbf{M}_2$ are denoted by F_1', F_2', and F_3' with $u_\alpha' \in F_\alpha' \cap F_{\alpha+1}'$ for each $\alpha = 1, 2, 3$. We illustrate this definition for the map shown in Fig. 8.

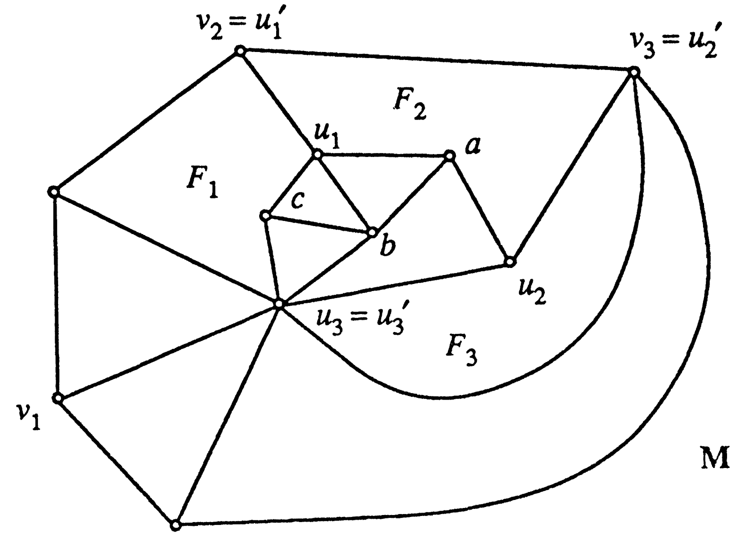

FIG. 7

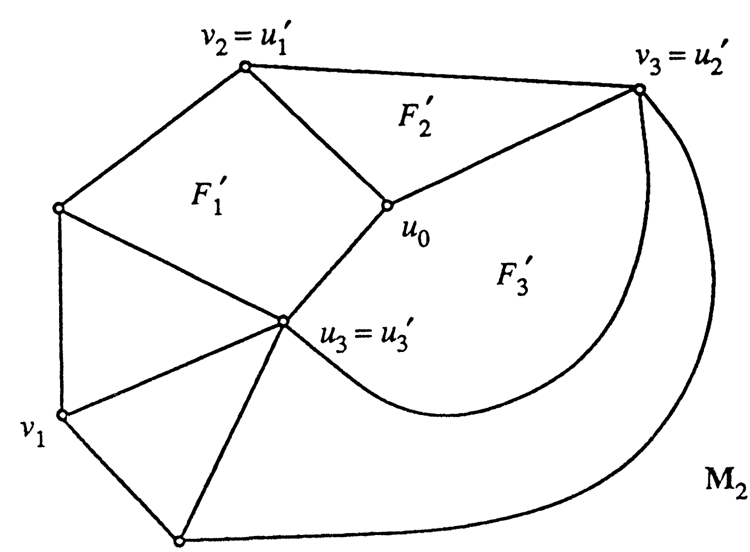

FIG. 8

We now show that (v_1, v_2, v_3) is a triad for $\mathbf{M}_2$ and that $\mathbf{M}_2$ satisfies the star-property for (v_1, v_2, v_3). It is obvious that (v_1, v_2, v_3) is a triad for $\mathbf{M}_2$ if F_1, F_2, and F_3 are interior faces. Now suppose that one of them, say F_3, is the exterior face. In this case, the path $\mathbf{M}[u_2', u_3']$ is a portion of the boundary of $\mathbf{M}$. In $\mathbf{M}_2$, this path is replaced by $u_2' \oplus (u_2, u_0, u_3) \oplus u_3'$, so that (v_1, v_2, v_3) is also a triad for $\mathbf{M}_2$.

Next, we show that $\mathbf{M}_2$ satisfies the star-property for (v_1, v_2, v_3). To the contrary, suppose that there exists a vertex $x \in V_2 - \{v_1, v_2, v_3\}$ for which there are two vertices y, z in $V_2 - \{x\}$ that separate x from $\{v_1, v_2, v_3\}$ in $\mathbf{M}_2$.

First, consider the case where $x = u_0$. Choose $\alpha \in \{1, 2, 3\}$ so that $u'_\alpha \notin \{y, z\}$. Clearly, $u'_\alpha \notin \{v_1, v_2, v_3\}$, so that in $\mathbf{M}$, there exist paths P_1, P_2, P_3, so that P_β is a path from u'_α to v_β and $P_\beta \cap P_{\beta+1} = \{u'_\alpha\}$ for each $\beta = 1, 2, 3$. Since $y, z \in V - \{u'_\alpha\}$, at least one of these paths, say P_γ, misses y and z in $\mathbf{M}$. If P_γ is a path in $\mathbf{M}_2$, we are done. Otherwise, P_γ contains at least two elements of $\{u_1, u_2, u_3\}$. Let P'_γ be the terminal segment of P_γ beginning with the last occurrence of an element of $\{u_1, u_2, u_3\}$ in P_γ. Then $u_0 \oplus P'_\gamma$ is a path from u_0 to v_γ in $\mathbf{M}_2 - \{y, z\}$.

Next, consider the case where $x \in V - \{u'_1, u'_2, u'_3\}$. Since $\mathbf{M}$ satisfies the star-property for (v_1, v_2, v_3), there exist paths P_1, P_2, P_3, so that P_α is a path from x to v_α and $P_\alpha \cap P_{\alpha+1} = \{x\}$ for each $\alpha = 1, 2, 3$. Any one of these three paths that is not a path in $\mathbf{M}_2$ must contain at least two elements of $\{u_1, u_2, u_3\}$, so at least two of P_1, P_2, and P_3 are paths in $\mathbf{M}_2$. So we may assume that P_α and $P_{\alpha+1}$ are paths in $\mathbf{M}_2$ with $y \in P_\alpha$ and $z \in P_{\alpha+1}$. We may also assume that $P_{\alpha+2}$ contains at least two elements from $\{u_1, u_2, u_3\}$. Let u_β be the first element from this set that belongs to $P_{\alpha+2}$ and let u_γ be the last. Then replace the portion of $P_{\alpha+2}$ beginning at u_β and ending with u_γ with (u_β, u_0, u_γ) to obtain a path from x to $v_{\alpha+2}$ in $\mathbf{M}_2 - \{y, z\}$.

Now suppose $x \in \{u'_1, u'_2, u'_3\}$. If neither y nor z is u_0, then y and z are vertices in $\mathbf{M}$, so there is a path P in $\mathbf{M}$ from x to $\{v_1, v_2, v_3\}$, with P avoiding y and z. If P is a path in $\mathbf{M}_2$, we are done. So we conclude that P contains at least two vertices from $\{u_1, u_2, u_3\}$. Let u_α be the last vertex from $\{u_1, u_2, u_3\}$, which belongs to P, and let P' be the terminal segment of P beginning at u_α. Then $(x, u_0, u_\alpha) \oplus P'$ is a path from x to $\{v_1, v_2, v_3\}$ in $\mathbf{M}_2$, which avoids y and z.

It remains only to consider the case where $x = u'_\alpha$ and one of the separating vertices, say y, is equal to u_0. If x has a neighbor w in $\mathbf{M}_2 - \{y, z, u'_1, u'_2, u'_3\}$, then we have that there is a path P in $\mathbf{M}_2$ from w to $\{v_1, v_2, v_3\}$ avoiding y and z. Then $(x, w) \oplus P$ is the desired path in $\mathbf{M}_2$. On the other hand, if x has no neighbor in $\mathbf{M}_2$ outside the set $\{y, z, u'_1, u'_2, u'_3\}$, then it is adjacent to one of the other $u'_1 \notin \{y, z\}$, say u'_β. Now if u'_β has a neighbor w in $\mathbf{M}_2 - \{y, z, u'_1, u'_2, u'_3\}$, then again there is a path P from w to $\{v_1, v_2, v_3\}$ in $\mathbf{M}_2$ avoiding y and z, which yields a path $(x, u'_\beta) \oplus P$ as required. Finally, if u'_β also has no neighbor outside $\{y, z, u'_1, u'_2, u'_3\}$, then the two vertices z and u'_γ, where $\gamma \notin \{\alpha, \beta\}$, separate u'_α (and also u'_β) from $\{v_1, v_2, v_3\}$ in $\mathbf{M}$, a contradiction.

This completes the argument that $\mathbf{M}_2$ satisfies the star-property for $\{v_1, v_2, v_3\}$. Now let $\mathcal{F}_2$ be a normal family of paths in $\mathbf{M}_2$ for the triad (v_1, v_2, v_3). We may assume without loss of generality that $u'_\alpha \in P(u_0, v_\alpha)$ for each $\alpha = 1, 2, 3$. We use $\mathcal{F}_0$ and $\mathcal{F}_2$ to construct a normal family for $\mathbf{M}$ as follows. Let $x \in V$. If $x \in V_0$, set $P(x, v_\alpha) = P_0(x, u_\alpha) \oplus P_2(u'_\alpha, v_\alpha)$. If $x \in V - V_0$, set $P(x, v_\alpha) = P_2(x, v_\alpha)$ when $u_0 \notin P_2(x, v_\alpha)$. If $x \in V - V_0$ and $u_0 \in P_2(x, v_\alpha)$, choose the unique elements u'_β, u'_γ for which u'_β precedes u_0 and u'_γ follows u_0 in $P_2(x, v_\alpha)$. Replace this portion of the path by $u'_\beta \oplus P_0(u_\beta, u_\gamma) \oplus u'_\gamma$. Verification that the resulting family of paths is normal is straightforward. Accordingly, we will assume in what follows that $\mathbf{M}$ does not contain a ring.

Case 5. We now present the closing argument.

Let v'_1 be the second vertex on the path $\mathbf{M}[v_1, v_2]$. Let $\mathbf{M}_0$ be the submap of $\mathbf{M}$ obtained by deleting v_1. On the path $P = \mathbf{M}_0[v_3, v'_1]$, rub out all vertices of degree 2 that are strictly between the end points of P. Call the resulting map $\mathbf{M}_1$.

Suppose there is a face G interior to $\mathbf{M}_1$ whose intersection with P does not form a single subpath. Let y and z be distinct vertices of P on G, with y on $\mathbf{M}_0[v_3, z]$, such that $\mathbf{M}_0[y, z] \cap G = \{y, z\}$.

There are two paths from z to y around the face G. Let P_0 be the one "nearer" to $\mathbf{M}_0[y, z]$, and let T be the region bounded by P_0 and $\mathbf{M}_0[y, z]$. This region has nonempty interior and contains some vertex w of $\mathbf{M}$ other than y and z, since $\mathbf{M}$ has no multiple edges. Clearly, $w \notin \{v_1, v_2, v_3\}$, so there is a path from w to $\{v_1, v_2, v_3\}$ in $\mathbf{M}$ avoiding $\{y, z\}$. There is no such path in $\mathbf{M}_1$, so this path must go to v_1 via an edge from some vertex strictly between y and z on $\mathbf{M}_0$.

Let $w_1, \dots, w_k$ be the vertices strictly between y and z on P, in the order they occur on P: we have just shown that $k \geq 1$. If $k = 1$, let F_1 and F_2 be the two faces incident with the edge $v_1 w_1$ in $\mathbf{M}$. Then (F_1, F_2, G) forms a ring, with the cycle C being the boundary of T, contradicting our assumption that $\mathbf{M}$ has no ring. If $k > 1$, let F_1 be the face incident with edge $v_1 w_1$ and not including w_2; and let F_2 be the face incident with $v_1 w_k$ and not including w_{k-1}. Again, (F_1, F_2, G) forms a ring, with the cycle C consisting of the boundary of T, with $\mathbf{M}_0[w_1, w_k]$ replaced by the two edges $w_1 v_1$ and $v_1 w_k$. Again, this is a contradiction, so there is no such face G.

In particular, $\mathbf{M}_1$ has no multiple edges.

Now it is easy to see that (v_1', v_2, v_3) is a triad for $\mathbf{M}_1$. We next show that $\mathbf{M}_1$ satisfies the star-property for (v_1', v_2, v_3); suppose not. Choose $x \in V_1 - \{v_1', v_2, v_3\}$ for which there exist two vertices $y, z \in V_1 - \{x\}$ that separate x from $\{v_1', v_2, v_3\}$ in $\mathbf{M}_1$. Since $\mathbf{M}$ satisfies the star-property, there exists a path P' from x to one of $\{v_1, v_2, v_3\}$ with P' missing y and z. It is obvious that P' terminates at v_1. Thus P' contains a vertex w from the path $P = \mathbf{M}_0[v_3, v_1']$.

Hence y and z both lie on P, one either side of w. We may suppose that $y \in \mathbf{M}_1[v_3, w]$ and $z \in \mathbf{M}_1[w, v_1']$. If y and z do not share a face inside $\mathbf{M}_1$, then $\mathbf{M}_1 - y - z$ is connected, a contradiction. Thus, y and z do share a face G inside $\mathbf{M}_1$, which therefore contains the whole of $\mathbf{M}_1[y, z]$. So w lies on $\mathbf{M}_1[y, z]$ and has degree 2 in $\mathbf{M}_1$, a contradiction.

Thus, $\mathbf{M}_1$ satisfies the star-property for (v_1', v_2, v_3). Now let $\mathcal{F}_1$ be a normal family of paths in $\mathbf{M}_1$ for (v_1', v_2, v_3). We construct a family $\mathcal{F}$ of paths in $\mathbf{M}$ as follows:

1. For every vertex $x \in V_1$, $P(x, v_\alpha) = P_1(x, v_\alpha)$ for $\alpha = 2, 3$.
2. For every vertex $x \in V_1$, let y be the first vertex on $P_1(x, v_1')$, which is adjacent to x in M and let $P_1(x, y)$ be the initial segment of this path ending at g. Then set $P(x, v_1) = P_1(x, y) \oplus v_1$.
3. For every vertex $\in V - V_1$ with $x \neq v_1$, set $P(x, v_1) = (x, v_1)$, $P(x, v_2) = \mathbf{M}_0[x, v_2]$ and $P(x, v_3) = \mathbf{M}_0^r[x, v_3]$.

It is an easy exercise to verify that $\mathcal{F}$ is a normal family of paths. This completes the proof of Lemma 4.1. □

5. The lower bound. For the sake of completeness, we include a proof of the following result, which is also proved in [7].

THEOREM 5.1. *If* $\mathbf{M}$ *is a convex polytope in* $\mathbb{R}^3$, *then* $\dim(\mathbf{P_M}) \geq 4$.

Proof. Suppose to the contrary that $\dim(\mathbf{P_M}) \leq 3$. Choose linear extensions L_1, L_2, L_3 of $\mathbf{P_M}$, so that $\mathbf{P_M} = L_1 \cap L_2 \cap L_3$. Of all the faces, let F_0 be the L_3-least. Then let $x_1, x_2, \dots, x_t$ be the vertices of F_0 and let $G_1, G_2, \dots, G_t$ be the faces that share an edge with F_0. We may assume that these vertices and faces have been labeled so that $x_i \in G_i \cap G_{i+1}$ for $i = 1, 2, \dots, t$.

Now $x_i < F_0 < G_j$ in L_3 for each i, j with $1 \leq i, j \leq t$. However, the subposet $\mathbf{P}_0$ of $\mathbf{P_M}$ generated by $\{x_i : 1 \leq i \leq t\} \cup \{G_i : 1 \leq i \leq t\}$ is isomorphic to a three-dimensional crown. The linear extension L_3 reverses no critical pairs of $\mathbf{P}_0$, which means they must all be reversed by L_1 and L_2. Since $\dim(\mathbf{P}_0) = 3$, this is impossible. □

Note that this argument actually shows that the subposet of $\mathbf{P}_\mathbf{M}$ consisting of vertices and faces has dimension at least 4.

6. Irreducible posets and duality. For $t \geq 2$, a poset $\mathbf{P}$ is said to be t irreducible if $\dim(\mathbf{P}) = t$ and $\dim(\mathbf{P} - x) < t$ for every $x \in \mathbf{P}$. The only 2-irreducible poset is a 2-element antichain. In [5] and [17], the collection of all 3-irreducible posets is determined. The posets in this collection can be grouped into seven infinite families with an additional eleven sporadic examples. For $t \geq 4$, constructions of t-irreducible posets are given in [4], [8], [9], and [14].

We find it interesting to note that each convex polytope in $\mathbb{R}^3$ determines a 4-irreducible poset in a natural manner.

THEOREM 6.1. *Let* $\mathbf{M}$ *be a convex polytope in* $\mathbb{R}^3$ *and let* F_0 *be an arbitrary face of* $\mathbf{M}$. *Then the subposet* $\mathbf{Q}_0 = \mathbf{P}_\mathbf{M} - \{F_0\}$ *is three-dimensional.*

Proof. Consider a plane drawing of the map $\mathbf{M}$ so that F_0 is the exterior face. Choose vertices v_1, v_2, v_3 on F_0 so that (v_1, v_2, v_3) is a triad. Then let $\mathcal{F}$ be a normal family of paths for (v_1, v_2, v_3).

Now consider the critical pairs in $\mathbf{Q}_0$. In addition to the Type 1 critical pairs of the form (y, F), where F is an interior face and $y \notin F$, we also have Type 2 critical pairs of the following form.

Type 2: (x, e), where x is a vertex on an interior face F, e is an edge common to F and the exterior face F_0, and x is not an end point of e.

Let L_1, L_2, and L_3 be the linear orders on V defined in the proof of Theorem 3.6. Extend L_1, L_2, and L_3 to linear extensions of $\mathbf{Q}_0$ by inserting the edges and faces as low as possible in each of the three orders. Call the resulting orders L_1^*, L_2^*, L_3^*. We show $\mathbf{Q}_0 = L_1^* \cap L_2^* \cap L_3^*$. It suffices to show that each Type 2 critical pair is reversed in some L_i^*. (We know from 3.6 that the Type 1 critical pairs are automatically reversed.)

Let (x, e) be a Type 2 critical pair. Let y and z denote the two end points of e. Choose α so that $F \subseteq S(x, \alpha)$. Then $y, z \in \mathbf{M}[v_{\alpha+1}, v_{\alpha+2}]$, $\emptyset = S(y, \alpha) = S(z, \alpha) \subsetneq S(x, \alpha)$. So it follows that (y, x) and (z, x) belong to Q_α. Thus, $x > e$ in L_α^*. □

When $\mathbf{M}$ is a planar 3-connected map, the planar dual $\mathbf{M}^d$ of $\mathbf{M}$ is also 3-connected. Furthermore, it is easy to see that the poset associated with the dual of $\mathbf{M}$ is the dual of the poset associated with $\mathbf{M}$. With this observation, we obtain the following dual form of the preceding theorem as well as the corollary summarizing the net effect of the two.

THEOREM 6.2. *Let* $\mathbf{M}$ *be a convex polytope in* $\mathbb{R}^3$ *and let* x *be an arbitrary vertex of* $\mathbf{M}$. *Then the subposet* $\mathbf{Q}_1 = \mathbf{P}_\mathbf{M} - \{x\}$ *is three-dimensional.*

COROLLARY 6.3. *Let* $\mathbf{M}$ *be a convex polytope in* $\mathbb{R}^3$. *Then the subposet of* $\mathbf{M}$ *determined by the vertices and faces is* 4*-irreducible.*

7. Concluding remarks. As mentioned earlier, we have been able to establish the upper bound $\dim(\mathbf{P}_\mathbf{M}) \leq 4$, on the dimension of $\mathbf{P}_\mathbf{M}$ when $\mathbf{M}$ is an arbitrary planar map. In the most general setting, we allow disconnected maps, loops, and multiple edges. However, we do not have an independent proof of this result. Our argument depends heavily on having the results and techniques of this paper in hand.

It is perhaps interesting to note here that the analogue of Theorem 6.1 does not hold for general planar maps. In the map M shown below (see Fig. 9), each critical pair (x_i, F_i) must be reversed in a different linear extension of P_M.

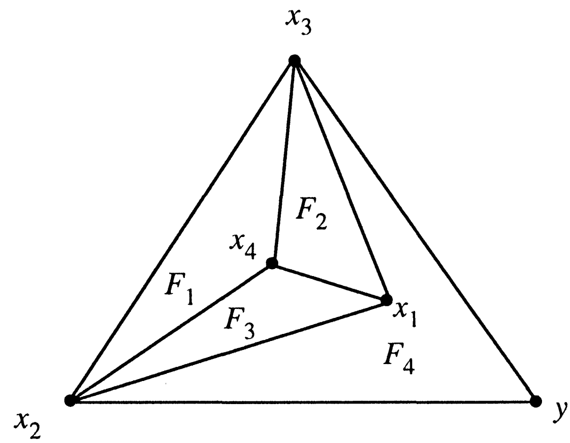

FIG. 9

It is relatively straightforward to show that for maps drawn on a surface of genus n, there is an upper bound of the form $\dim(\mathbf{P_M}) \leq f(n)$. It would be of some interest to determine $f(n)$. Perhaps the correct answer is $f(n) = n + 4$.

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