6. D. J. KLEITMAN, Families of Non-disjoint Subsets, *Journal of Combinatorial Theory* **1**, (1966) 153-155.

7. F. P. RAMSAY, Collected papers, pp. 82–111; see also reference 2.

8. P. TURAN, On the Theory of Graphs, *Colloq. Math.* **3** (1954), 19–30.

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Families of Non-disjoint Subsets*

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A family F of subsets of a finite set which contains no two disjoint subsets can contain at most half of all the subsets, since no subset and its complement can be in F. Moreover, if F is maximal with respect to this property (so that any larger family does not satisfy it) F must contain exactly half the subsets. (If F is maximal, $B \in F$, $C \supset B$ implies $C \in F$; also, $B \notin F$ implies that there is a $C \in F$ such that $C \cap B = \varphi$, i.e., $C \subset \bar{B}$ ($\bar{B}$ is the complement of B) hence $\bar{B} \in F$. Thus B or $\bar{B}$ must be in F.) In this paper we consider the analogous limitations on the number of subsets contained in k disjoint families $F_1, \ldots, F_k$ each of which contains no disjoint subsets. We prove the following result, which was conjectured by Erdös (private communication).

THEOREM. *If $F_1, \ldots, F_k$ are families of subsets of an n element set such that $A_i \cap A_j \neq \varphi$ if $A_i, A_j \in F_l$ for $1 \leq l \leq k$, then the number of elements in the union of $F_1, \ldots, F_k$ is no greater than $2^n - 2^{n-k}$:*

$$\left|\bigcup_{j=1}^{k} F_j\right| \leq 2^n - 2^{n-k}$$

(*where $|A|$ denotes the number of elements of A*).

Unlike the result for one family, the minimum number of subsets in the union of k disjoint F's which are maximal with respect to these

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properties is not the maximum $(2^n - 2^{n-k})$. We can in fact construct k disjoint families to none of which can a subset not already contained in one be added, whose union contains as few as

$$2^{n-1} + \sum_{j=1}^{k-1} \binom{n-1}{j-1}$$

subsets. The union of F nondisjoint families each maximal with respect to our property can contain as few as $2^{n-1} + l$ subsets with $2^l \geq k$ for

$$k \leq 2\binom{n}{\left[\frac{n}{2}\right]+1}.$$

The theorem follows directly from the following lemma.

LEMMA: *Let U be a family of subsets of an n element set such that $A \in U$, $B \supset A$ implies that $B \in U$, and let L be a family of subsets of the same set such that $A \in L$, $B \subset A$ implies that $B \in L$. Then*:

$$|U \cap L|\, 2^n \leq |U|\, |L|.$$

PROOF: The result is trivial for $n = 1$. We suppose it to be true for $n = k$, and consider a pair of families U and L of a $k + 1$ element set S. If **a** is any element of S we can write, with U_a, $U_{\bar{a}}$, L_a, $L_{\bar{a}}$ disjoint:

$$U = U_a \cup U_{\bar{a}}$$
$$L = L_a \cup L_{\bar{a}}$$

where U_a and L_a consist of the subsets in U and L, respectively, which contain **a**. By the defining properties of U and L we have $[U_a] \geq [U_{\bar{a}}]$, $[L_a] \leq [L_{\bar{a}}]$ since, for example, the union of any element of $U_{\bar{a}}$ with $\{a\}$ must be in U_a. Moreover, $U_{\bar{a}}$ and $L_{\bar{a}}$ are families of subsets of $S - \{a\}$ with the same properties as U and L and so must satisfy

$$|U_{\bar{a}} \cap L_{\bar{a}}|\, 2^k \leq |U_{\bar{a}}|\, |L_{\bar{a}}|. \tag{1}$$

The families U_a' and L_a' which consist of the subsets obtained by removing **a** from the subsets in U_a and L_a are likewise families of subsets of $S - \{a\}$ with the properties of U and L, so that

$$|U_a \cap L_a| \, 2^k = |U_a' \cap L_a'| \, 2^k \leq |U_a'| \, |L_a'| = |U_a| \, |L_a|. \tag{2}$$

Combining relations (1) and (2) we obtain

$$\begin{aligned} 2^{k+1}|U \cap L| &= 2^{k+1}(|U_a \cap L_a| + |U_{\bar{a}} \cap L_{\bar{a}}|) \\ &\leq 2 \, (|U_a| \, |L_a| + |U_{\bar{a}}| \, |L_{\bar{a}}|). \end{aligned} \tag{3}$$

Since $|U_{\bar{a}}| \geq |U_a|$ and $|L_a| \leq |L_{\bar{a}}|$ we have

$$|U_a| \, |L_a| + |U_{\bar{a}}| \, |L_{\bar{a}}| \leq |U_a| \, |L_{\bar{a}}| + |U_{\bar{a}}| \, |L_a|. \tag{4}$$

Combining relations (3) and (4) yields the desired result.

Proof of our theorem follows from this lemma by induction on k. Let $F_1, ..., F_k$ be families of non-disjoint subsets of S; extend each arbitrarily to a maximal family F'. Each maximal family F_j' (and any union of such) contains any subset which contains any subset already in it. Let U be $U_{s=1}^{k-1} F_j$. Let L be $\bar{F}_k'$. Since F_k' is maximal, $|L| = |F_k'| = 2^{n-1}$. By hypothesis $[U] \leq 2^n - 2^{n-k+1}$. From our lemma, then,

$$\begin{aligned} | \bigcup_{j=1}^{k} F_j| = |U \cup F_k| \leq |F_k'| + |U \cap L| &\leq |F_k'| + 2^{-n}|U| \, |L| \\ &\leq 2^n - 2^{n-k}, \end{aligned}$$

which proves the theorem.

If we choose F_j to be all subset of S which contain **a** and $(j-1)$ other elements, and all $(n-j+1)$ element subsets of S which do not contain **a**, for $j \leq k$ we find that

$$2^{n-1} + \sum_{j=0}^{k-2} \binom{n-1}{j}$$

subsets contain subsets that are in some F's. If the set of F's are then extended to be disjoint and maximal, they will contain this many subsets.

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