2

Chains and Antichains

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2.1 Chapter Overview

Combinatorial problems involving chains and antichains are at the heart of research on posets. In this chapter, we discuss classic theorems such as Dilworth's theorem and its links to other linear programming gems. We also provide proofs of the theorems of Greene and Kleitman, with a special emphasis on the duality between the two results. We apply these results back to the Erdős-Szekeres theorem for monotonic sequences, and we close the chapter with a concise treatment of quite recent results by Duffus-Sands and Howard-Trotter concerning pairwise disjoint families of maximal chains and antichains.

2.2 Dilworth's Theorem and Its Dual

Perhaps the two most fundamental parameters of partially ordered sets are *height* and *width*. These parameters measure, quite naturally enough, how "tall" and how "broad" the poset is. The *height* $h(P)$ of a poset† P is the maximum cardinality of a chain in P‡, while the *width* $w(P)$ of P is the maximum cardinality of an antichain in P. Of course, if P is finite, then both its height and width are finite, and as will be clear from the material to follow, there is a natural duality between these two parameters.

Now consider the following natural extremal problem: Given a poset P,

† As we commented in Chapter 1, when a poset remains fixed throughout a discussion, researchers often prefer to use a single symbol such as P to denote a poset rather than the full $P = (X, P)$ and that is the convention we will follow in this chapter.

‡ Caution: Some authors use the term "height" for what is in our notation $h(P)-1$, so that the two-element chain **2** has height 1. We are convinced that our convention is, for most purposes, the more appropriate. To avoid any possibility of confusion, we never speak of the "length" of a chain or a poset, which should properly be taken as the height minus one.

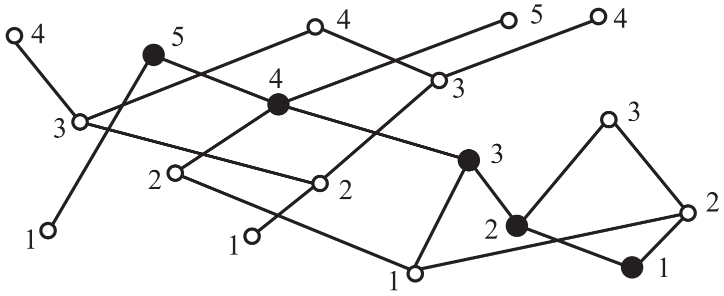

Fig. 2.1. A Poset of Height 5

find the least positive integer s for which there is a partition

$$P = A_1 \cup A_2 \cup \dots A_s$$

of P into pairwise disjoint subsets $A_1, A_2, \dots, A_s$ with A_j an antichain for each $j = 1, 2, \dots, s$. Evidently, s is at least as large as the height of P. Not surprisingly, s is exactly equal to the height of P. The resulting elementary theorem is generally credited to Mirsky [99], but it was known as part the folklore of the subject and in fact appears as a remark in [99].

Theorem 2.2.1 *A poset P of height h has a partition $P = A_1 \cup A_2 \cup \dots A_h$ with A_j an antichain, for each $j = 1, 2, \dots, h$.*

Proof For each $x \in X$, let $h(x)$ be the largest integer r for which there exists a chain $x_1 < x_2 < \cdots < x_r$ with $x = x_r$. Evidently, $h(x) \leq h$ for all $x \in P$. Then for each $j = 1, 2, \dots, h$, let $A_j = \{x \in X : h(x) = j\}$. It is easy to see that each A_j is an antichain. $\square$

This proof provides an efficient algorithm for finding a maximum chain and a minimum size partition into antichains. Note that A_1 is just the set of minimal elements of P. Thereafter, A_{j+1} is the set of minimal elements of what remains when A_1, $A_2, \dots, A_j$ have been removed. Furthermore, for each j with $1 \leq j < h$, if $x \in A_{j+1}$, then there exists some $y \in A_j$ with $x < y$ in P. So we can find a maximum chain in P by backtracking. This approach is illustrated in Figure 2.2, where we show a partition of P into 5 antichains. The black points form a chain of size 5.

Now the dual problem: Given a poset P, find the least positive integer t for which there is a partition

$$P = C_1 \cup C_2 \cup \dots C_t$$

of P into pairwise disjoint subsets $C_1, C_2, \ldots, C_t$ with C_i a chain for each $i = 1, 2, \ldots, t$. Evidently, t is at least as large as the width of P, and the classic theorem of R. P. Dilworth [99] asserts that these two values are equal.

Theorem 2.2.2 *A poset P with* $\text{width}(P) = t$ *has a partition* $P = C_1 \cup C_2 \cup \ldots C_t$ *with C_i a chain, for each* $i = 1, 2, \ldots, t$.

We will provide four (yes, four!) proofs of Dilworth's theorem, starting with the one found in most texts. Here, the presentation is simplified by the following notation. When P is a poset and $x \in P$, we let $D(x) = \{y \in P : y < x \text{ in } P\}$; $D[x] = \{y \in P : y \leq x \text{ in } P\}$; $U(x) = \{y \in P : y > x \text{ in } P\}$; $U[x] = \{y \in P : y \geq x\}$; and $I(x) = \{y \in P - \{x\} : x \| y \text{ in } P\}$. When A is a maximal antichain in P, we let $D(A) = \{y \in P : y < a \text{ in } P$, for some $a \in A\}$ and $D[A] = A \cup D(A)$. The subsets $U(A)$ and $U[A]$ are defined analogously.

Proof We proceed by induction on $|P|$, the result being trivial if $|P| = 1$. Assume validity for all posets with $|P| \leq k$ and suppose that P is a poset with $|P| = k + 1$. Choose a maximal point x and a minimal point y with $y \leq x$ in P. Then set $C = \{x, y\}$, noting that $|C| = 1$ if $x = y$. Regardless, $|P - C| \leq k$.

Let w' denote the width of the subposet $P - C$. If $w' < w$, then there is a partition $P - C = C_1 \cup C_2 \cup \cdots \cup C_{w'}$ with each C_i a chain. This implies that $P = C \cup C_1 \cup C_2 \cup \ldots C_{w'}$ is a partition of P into $w' + 1$ chains. Since $w' + 1 \leq w$, the result holds for P.

Finally, we are left to consider the case where $w' = w$. Choose a w-element antichain $A = \{a_1, a_2, \ldots, a_w\} \subseteq P - C$. Note that $x \in U(A)$, $x \notin D[A]$, $y \in D(A)$ and $y \notin U[A]$. Therefore, we can partition the subposets determined by $U[A]$ and $D[A]$, respectively, into w chains.

Without loss of generality, we may label these partitions as $U[A] = C_1' \cup C_2' \cup \cdots \cup C_w'$ and $D[A] = C_1'' \cup C_2'' \cup \cdots \cup C_w''$, where $a_i \in C_i' \cap C_i''$ for $i = 1, 2, \ldots, w$. However, this implies that $P = (C_1' \cup C_1'') \cup (C_2' \cup C_2'') \cup \cdots \cup (C_w' \cup C_w'')$ is a partition of P into w chains. □

We illustrate Dilworth's theorem with a diagram for a poset of width 7 in Figure 2.2. The diagram also shows a partition into 7 chains, and the black points form an antichain of size 7. Note that the chains need not consist of "contiguous" points in the diagram.

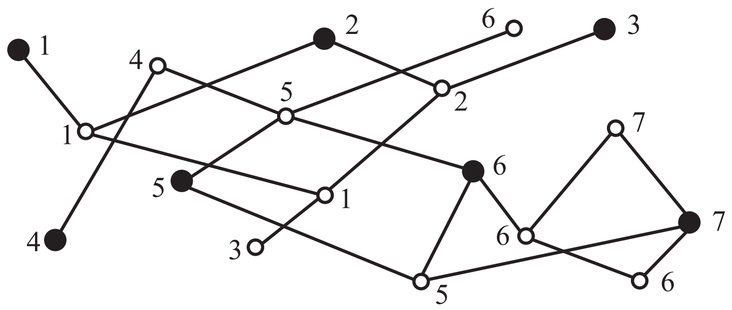

Fig. 2.2. A Poset of Width 7

2.3 Network Flows

Many undergraduate texts group Dilworth's theorem with other classic combinatorial theorems, all exhibiting a common linear-programming theme. In order that the arguments to follow will make sense, we provide a concise summary of these results and how they are proved. Many readers can give this section and the one to follow little more than a quick glance.

We consider *networks*, weighted digraphs satisfying the following three properties: (1) There are distinguished vertices S and T called respectively, the *source* and *sink*; (2) All edges incident with S are oriented away from S, and all edges incident with T are oriented towards T; and (3) For every edge (x, y), there is a non-negative capacity $c(x, y)$.

A *flow* ϕ in a network is a function which assigns to each directed edge $e = (x, y)$ a non-negative value $\phi(e) = \phi(x, y) \leq c(x, y)$ so that the following "conservation" law holds:

Amount In Equals Amount Out. For every vertex x which is neither the source nor the sink, $\sum_u \phi(u, x) = \sum_v \phi(x, v)$, i.e., the amount entering x is equal to the amount exiting x.

The *value* of a flow ϕ is the quantity $\sum_x \phi(S, x)$, which by the conservation law is also the quantity $\sum_x \phi(x, T)$. In Figure 2.3, we illustrate a network flow of value 84. In this figure, edges carry two labels. The first is the capacity of the edge, and the second is the flow on the edge.

Of course, a network flow models the movement of goods from a point of origination to a terminal point, subject to constraints on how much can be transported across individual edges. The natural combinatorial problem is then to find a flow of maximum value. Clearly, this is a linear-programming problem, but there are special purpose algorithms which perform better than an all-purpose LP algorithm in finding maximum flows.

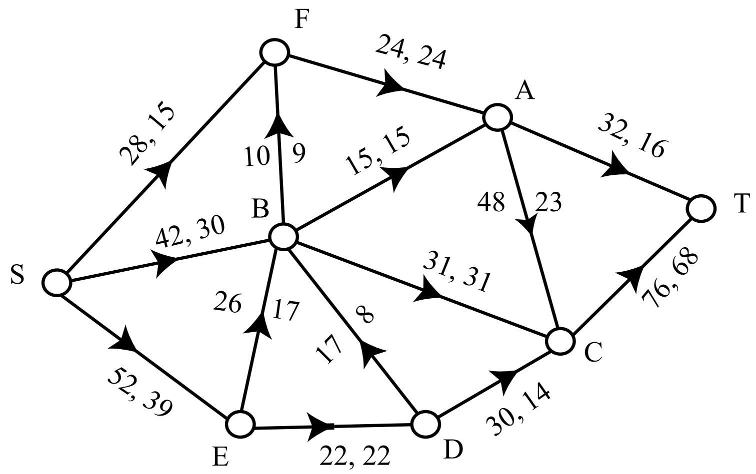

Fig. 2.3. A Network Flow Problem

2.3.1 Flows and Cuts

A *cut* in a network is a $(\mathcal{L},\mathcal{U})$ where $\mathcal{L}\cup\mathcal{U}$ is a partition of the vertex set into two non-empty subsets with $S\in\mathcal{L}$ and $T\in\mathcal{U}$. The *capacity* of a cut $(\mathcal{L},\mathcal{U})$, denoted $c(\mathcal{L},\mathcal{U})$, is defined by

$$c(\mathcal{L},\mathcal{U}) = \sum_{x\in\mathcal{L},y\in\mathcal{U}} c(x,y).$$

The conservation law implies the following inequality: If ϕ is a flow and $(\mathcal{L},\mathcal{U})$ is a cut, then the value of ϕ cannot exceed the capacity $c(\mathcal{L},\mathcal{U})$ of the cut. Not surprisingly, this basic inequality is tight.

Theorem 2.3.1 *The maximum value of a flow in a network is equal to the minimum capacity of a cut.*

Although we do not include the full details of the proof of this theorem here, we sketch the key ideas as they are central to arguments which follow. Let ϕ be a flow in a network, and let x be a vertex distinct from the source S. An *augmenting path* from S to x is a sequence $(S=u_0,u_1,u_2,\ldots,u_m=x)$ of distinct vertices so that for each $i=0,1,\ldots,m-1$, either (1) The network contains the edge (u_i,u_{i+1}) and $\phi(u_i,u_{i+1})<c(u_i,u_{i+1})$ (a *forward* edge), or (2) the network contains the edge (u_{i+1},u_i) and $\phi(u_{i+1},u_i)>0$ (a *backward* edge). When there is an augmenting path from S to T, then the value of the flow can be increased by (1) increasing the flow on all forward edges and (2) decreasing the flow on all backwards edges.

If there are no augmenting paths from source to sink, then there is a natural cut $\mathcal{L} \cup \mathcal{U}$ where $\mathcal{L}$ consists of S together with all points reachable from S by augmenting paths. The conservation law now implies that the value of the flow is the capacity of this cut.

The Ford-Fulkerson labeling algorithm searches for an augmenting path from source to sink using breadth first search. This insures that the number of vertices in the augmenting path has, at each step, the minimum number of vertices. Paying attention to this detail means that the procedure will halt with the number of iterations bounded as a function of the number of vertices. Also, this algorithm implies that a network flow problem with integer capacities has a optimum solution with integer valued flows. In particular, if all capacities are 1, then there is an optimum flow in which the flow on every edge is either 0 or 1.

As an illustration, consider the augmenting path $P = (S, B, D, C, T)$ in the network shown in Figure 2.3. On this path, the edge (B, D) is backwards while the remaining edges are forward. The amount of increase is limited to 8 by the foward edge (C, T) where the capacity is 76 and the current flow is 68. Also, the backwards edge (B, D) limits the increase to 8 as there is only a current flow of 8 on this edge. After augmentation, the value of the flow is 92. Also, the labeling algorithm, when applied to the updated network, will halt with $\mathcal{L} = \{S, B, E, F\}$ and $\mathcal{U} = \{A, C, D, T\}$. The capacity of this cut is $24 + 15 + 31 + 22 = 92$.

2.4 Combinatorial Classics

Let $G = (A, B, E)$ be a bipartite graph. Then for each subset $S \subseteq X$, let $N(S) = \{b \in B : \text{there exists some } s \in S \text{ with } sb \in E\}$. Here is the first theorem in this grouping, popularly known as the defect form of Hall's marriage theorem.

Theorem 2.4.1 (Hall's Theorem (Defect Form)) *Let $G = (A, B, E)$ be a bipartite graph. Then the size of the largest matching in G is $|A| - d$ where d is the largest non-negative integer for which there exists a subset $S \subseteq A$ with $|N(S)| \leq |A| - d$. In particular, there a complete matching of A into B if and only if $|N(S)| \geq |S|$ for every subset $S \subseteq A$.*

Proof Here are the key ideas behind the proof. Clearly, no matching can be any bigger than $|A| - d$. We show that there is one of this size.

From the bipartite graph, we construct a network flow problem in which all edges have capacity 1. Attach a source S with an edge from S to a for each vertex $a \in A$. Attach a sink T with an edge from b to T for each $b \in B$. Orient all edges in G from A to B. Turn on a network flow algorithm and find a maximum integral valued flow. Edges from A to B having flow 1 are the edges in the maximum matching.

When the network flow algorithm halts with the partition $(\mathcal{L}, \mathcal{U})$, let $S = \mathcal{L} \cap A$ and let m be the number of vertices in S which are matched. Clearly, all vertices in A which are not matched belong to S. It follows that $|N(S)| = m$. □

In a similar vein, an analysis of the halting condition leads to the following theorem.

Theorem 2.4.2 (König/Egerváry Theorem) *Let $G = (A, B, E)$ be a bipartite graph. Then the maximum size of a matching in G is equal to the minimum number of vertices required to cover all edges of G.*

Theorem 2.4.3 (Menger's Theorem, Vertex Version) *Let G be a graph and let x and y be non-adjacent vertices in G. Then the minimum number of vertices whose removal from G leaves x and y in separate components is equal to the maximum number of vertex disjoint paths from x to y in G.*

Proof Again a sketch of the key ideas: Form a network flow problem with x as source and y as sink. All edges incident with x oriented away from x. All edges incident with y oriented towards y. All other edges uv replaced by two edges (u, v) and (v, u). Split each vertex u distinct from x and y into two new vertices u' and u''. All edges that were coming into u now go to u'. All edges that used to leave u now leave from u''. Add edge (u', u''). All edges in the resulting network have capacity 1. Turn on a max-flow algorithm. From each path in the resulting flow, find a point u where u' is labeled and u'' is not. This results in a separating set whose size is the number of paths from x to y. □

Theorem 2.4.4 (Menger's Theorem, Edge Version) *Let G be a graph and let x and y be distinct vertices in G. Then the minimum number of edges whose removal from G leaves x and y in separate components is equal to the maximum number of edge disjoint paths from x to y in G.*

Proof Same idea as the vertex version, but even easier, as we don't need to split vertices. □

When G is a graph and $S \subset G$, let $\text{odd}(S)$ denote the number of odd components in the induced subgraph $G - S$. Here's a theorem in this class whose proof (as typically presented) uses Hall's matching theorem at a key point. We state the result in defect form

Theorem 2.4.5 (Tutte's 1-Factor Theorem (Defect Form)) *Let $\mathcal{M}$ be a maximum matching in a graph G. The number of unmatched vertices equals the largest non-negative integer d for which there is a set S of vertices for which* $\text{odd}(S) = |S| - d$. *In particular, G has a perfect matching if and only if* $\text{odd}(S) \le |S|$ *for every subset S of the vertex set of G.*

2.4.1 The Gallai-Millgram Theorem

Although it belongs to the class of problems we discussed in the preceding section, we place the Gallai-Millgram theorem in a category by itself because of its role in the proof of the Greene-Kleitman theorem which will come later in this chapter. Let G be an oriented graph. A *directed path* P is a sequence $x_1, x_2, \ldots, x_t$ of distinct vertices of G so that (x_i, x_{i+1}) is an edge of G for all $i = 1, 2, \ldots, t-1$. The vertex x_1 is the *starting point* of P while x_t is the *ending point* of P. A *path partition* of G is a family $\mathcal{P}$ of directed paths in G so that every vertex of G belongs to exactly one path in $\mathcal{P}$. When $\mathcal{P}$ is a path partition of G, we let $E(\mathcal{P})$ denote the set of ending points of the paths in $\mathcal{P}$.

Here is Gallai-Millgram theorem [99], which we state in the "strong" form that facilitates an easy inductive proof—and it is this strong form that we will need later.

Theorem 2.4.6 (Gallai-Millgram Theorem) *Let G be an oriented graph, let m be the independence number of G, and let $\mathcal{P}$ be a path partition of G. Then there exists a path partition $\mathcal{Q}$ of G so that $|\mathcal{Q}| \le m$ and $E(\mathcal{Q}) \subseteq E(\mathcal{P})$.*

Proof We proceed by induction on the number of vertices in G. The theorem holds trivially when G is a graph with only one vertex. Now assume that for some $k \ge 1$, the theorem holds for all oriented graphs with at most k vertices, and suppose that G has $k+1$ vertices.

Let $\mathcal{P}$ be a path partition of G, and let m be the independence number of G. We may assume that $|\mathcal{P}| > m$, else we simply take $\mathcal{Q} = \mathcal{P}$. Now let $P \in \mathcal{C}$, let x be the ending point of P, let $G' = G - P$, and let m' be the independence number of G'. Note that $m' \le m$. By the inductive hypothesis, there exists a path partition $\mathcal{P}'$ of G' with $|\mathcal{P}'| \leq m'$ and $E(\mathcal{P}') \subseteq E(\mathcal{P}) - \{x\}$. Set $\mathcal{Q}' = \{C\} \cup \mathcal{P}'$.

If $|\mathcal{Q}'| < m$, then we may take $\mathcal{Q}$ to be $\mathcal{Q}'$. So we may assume that $m' = m$, so that $|\mathcal{Q}'| = m + 1$. Label the paths in $\mathcal{Q}'$ as $P_1, P_2, \ldots, P_{m+1}$. Also, for each $i = 1, 2, \ldots, m + 1$, let x_i be the ending point of P_i,

The set $E(\mathcal{Q}') = \{x_1, x_2, \ldots, x_{m+1}\}$ is not an independent set in G, so without loss of generality, we may assume that these paths have been labelled so that (x_2, x_1) is an edge in G. If $|P_1| = 1$, then we may take $\mathcal{Q}$ as $\{P_2 \cup \{x_1\}, P_2, P_3, \ldots, P_{m+1}\}$. So we may assume that $|C_1| \geq 2$. Let y_1 be the next to last point on P_1.

Now let $G'' = G - x_1$, and let m'' be the independence number of G''. Of course, $m'' \leq m$. We apply the inductive hypothesis to the path partition $\mathcal{P}'' = \{P_1 - \{x_1\}, P_2, \ldots, P_{m+1}\}$ of G'' and conclude that there exists a path partition $\mathcal{Q}''$ of G'' so that $|\mathcal{Q}''| \leq m''$ and $E(\mathcal{Q}'') \subseteq E(\mathcal{P}'') \subseteq E(\mathcal{P})$. If y_1 is the ending point of a path $P \in \mathcal{Q}''$, we simply add x_1 to the end of P and take the resulting path partition as $\mathcal{Q}$. So we may assume that $y_1 \notin E(\mathcal{Q}'')$.

If x_2 is the ending point of a path in $\mathcal{Q}''$, then we can add x_1 on the end of that path to form $\mathcal{Q}$. Finally, if $x_2 \notin \mathcal{Q}''$, then $|\mathcal{Q}''| < m$, so we can form $\mathcal{Q}$ by adding x_1 as a 1 point path. $\square$

When C is a chain in a poset P, we consider the largest element of C as its *ending point*. When $\mathcal{C} = \{C_1, C_2, \ldots, C_t\}$ is a chain partition, we let $E(\mathcal{C})$ denote the set of ending points of the chains in $\mathcal{C}$. The following stronger version of Dilworth's theorem now follows as an immediate corollary.

Corollary 2.4.7 (Dilworth's theorem) *If P is a poset of width w, and $\mathcal{C}$ is a chain partition of P, then there exists a chain partition $\mathcal{P}$ of P so that $|\mathcal{P}| = w$ and $E(\mathcal{P}) \subseteq E(\mathcal{C})$.*

We close this section with an easy corollary, which will be useful later in this chapter.

Corollary 2.4.8 *Let P be a poset and let S be an upset in P. If* $\mathrm{width}(S) = v < w = \mathrm{width}(P)$*, then there exists a chain partition $\mathcal{C} = \{C_1, C_2, \ldots, C_v, C_{v+1}, \ldots, C_w\}$ of P so that $C_i \cap S = \emptyset$ when $v < i \leq w$.*

Proof First let $\mathcal{D} = \{D_1, D_2, \ldots, C_v\}$ be a chain partition of S. Then form a chain partition of P by adding one point chains for each point in $X - S$. Then apply Corollary 2.4.7. $\square$

2.4.2 Algorithmic Issues for Dilworth's Theorem

Neither of the first two proofs we have provided for Dilworth's theorem seems to provide an effective algorithm for finding the width w of a poset and a partition into w chains. However, Fulkerson [99] pointed out that this apparent shortcoming can be readily overcome by exploiting the power of the max-flow/min-cut theorem. Here is an outline of how to proceed.

Let P be a poset, and let $|P| = n$. Form a bipartite graph $G = (X', X'', E)$ with $X' = \{x' : x \in P\}$, $X'' = \{x'' : x \in P\}$ and $E = \{x'y'' : x < y \text{ in } P\}$. Use the network flow algorithm discussed earlier in this chapter to find a maximum matching in G. Use this matching to construct a chain partition of P by the following rule: A point x is covered by a point y in the chain partition if and only if $x'y''$ is an edge in the maximum matching. In making this definition, we do not mean that covers in the chain partition are covers in the poset.

Now here is the first of two key properties concerning the resulting chain partition. The first involves the set of labeled and unlabeled points when the network flow algorithm halts. The elementary proof is left as an exercise.

Proposition 2.4.9 *If $C = \{x_1 < x_2 < \cdots < x_m\}$ is one of the chains constructed from the maximum matching, then there is some i for which x_i' is labeled and x_i'' is unlabeled.*

Here is the second observation. Again the elementary proof is omitted.

Proof Suppose the maximum matching results in w chains $C_1, C_2, \ldots, C_w$. Also suppose that for each $i = 1, 2, \ldots, w$, we have chosen a point x_i so that x_i' is labeled while x_i'' is not. Then $A = \{x_1, x_2, \ldots, x_w$ is an antichain. □

2.4.3 The Erdös-Szekeres Theorem

The following elementary result follows immediately from (either form of) Dilworth's theorem.

Lemma 2.4.10 *If P is a poset, s and t are non-negative integers and $|X| \geq st+1$, then either P contains a chain of size $s+1$ or an antichain of size $t+1$.*

The following classic result follows easily.

Corollary 2.4.11 (Erdös-Szekeres Theorem) *Let s and t be non-negative numbers. Then in any sequence of $st+1$ distinct real numbers, there is ei-*

ther an increasing subsequence of length $s+1$ or a decreasing subsequence of length $t+1$.

Proof Define a partial order P on the set of numbers in the sequence by setting $a_i < a_j$ in P if and only if $i < j$ in $\mathbb{Z}$ and $a_i < a_j$ in $\mathbb{R}$. A chain in P is an increasing subsequence while an antichain in P is a decreasing subsequence. $\square$

Here is a reformulation of the Erdős-Szekeres theorem, one that is has a number of applications. The proof is an easy induction on the parameter k.

Corollary 2.4.12 *Let m, k and n be positive integers. If $n > m^{2^k}$ and $L_0, L_1, \ldots, L_k$ is a family of linear orders on $[n]$, then there exists an m-element subset $S \subset [n]$ so that for each $j = 1, 2, \ldots, k$, either $L_j(S) = L_0(S)$ or $L_j(S) = L_0^d(S)$.*

2.5 Perfect Graphs and Comparability Graphs

Many of the results discussed earlier in this chapter are pieces of a more general topic: the theory of perfect graphs.

For a graph G, let $\chi(G)$ denote the chromatic number of G, and let $\omega(G)$ denote the clique number. Evidently $\chi(G) \geq \omega(G)$ for all G, and in fact, one the basic facts about chromatic number is that for every $t \geq 2$, there is a graph G with $\omega(G) = 2$ and $\chi(G) = t$. Nevertheless, there is particular interest in graphs for which $\chi(G) = \omega(G)$.

A graph G is *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G,

It is easy to see that an odd cycle C_{2n+1} with $2n+1 \geq 5$ vertices is *not* perfect, as it has clique number 2 and chromatic number 3. Furthermore, its complement $\overline{C_{2n+1}}$ is also not perfect, as it has clique number n and chromatic number $n+1$. In 1961, C. Berge made the following challenging conjecture:

Conjecture 2.1 (Strong Perfect Graph Conjecture) *A graph is perfect if and only if it contains no induced subgraph isomorphic to either an odd cycle C_{2n+1} ($n \geq 2$) or the complement of an odd cycle $\overline{C_{2n+1}}$ ($n \geq 2$).*

In 19xx, Lovász [99] proved the following intermediate result, which is usually referred to as the "Perfect Graph Theorem" with Berge's conjecture then called the "Strong Perfect Graph Conjecture."

Theorem 2.5.1 (Perfect Graph Theorem) *A graph G is perfect if and only if its complement $\overline{G}$ is perfect.*

Dilworth's theorem asserts that the complement of a comparability graph is perfect, while the dual form (partition into antichains) is equivalent to showing that comparability graphs are perfect. So in view of Lovász's Perfect Graph Theorem, either result follows immediately from the other. So here is our fourth and final proof of Dilworth's theorem. Since a poset of height h can be partitioned into h antichains, it follows that comparability graphs are perfect. So by Theorem 2.5.1, so are incomparability graphs. This implies Dilworth's theorem!

We close this brief section with the remark that in 2003, Chudnovsky, Robertson, Seymour and Thomas [99] succeeded in settling Berge's conjecture in the affirmative.

Theorem 2.5.2 (Strong Perfect Graph Theorem) *A graph is perfect if and only if it contains no induced subgraph isomorphic to either an odd cycle C_{2n+1} ($n \geq 2$) or the complement of an odd cycle $\overline{C_{2n+1}}$ ($n \geq 2$).*

The study of classes of perfect graphs, and the special case of comparability and incomparability graphs in particular, remains an important topic in graph theory, and we will return to the subject later in this monograph.

2.6 The Theorems of Greene and Kleitman

In this section, we shall discuss powerful extensions of Dilworth's Theorem and its dual, due to Greene and Kleitman [**?**]. Readers who have some familiarity with this topic will sense right from the outset that our treatment does not follow traditional lines. Instead, we elect to emphasize duality throughout. However, at key points, we will pause to make sure the connections between our approach and the original perspective is clear.

We start with some terminology. For the remainder of this chapter, we only refer to a collection $\mathcal{S}$ of sets as a *family* when the sets in $\mathcal{S}$ are non-empty and pairwise disjoint. When $|\mathcal{S}| = s$, we will $\mathcal{S}$ is a s-family. Note that s is just the *cardinality* of $\mathcal{S}$. We will denote by $\langle\mathcal{S}\rangle >$ the set $\bigcup\{S : S \in \mathcal{S}\}$, a notation we prefer to $\bigcup\mathcal{S}$. We will refer to $|\langle\mathcal{S}\rangle|$ as the *size* of $\mathcal{S}$. Naturally, we will be concerned primarily with the case where $\mathcal{S}$ is either a family of chains or a family of antichains in a poset P. However, in both cases, it will be important that we do not require that a family in P be a partition of P, i.e., in general $\langle\mathcal{S}\rangle$ will be a proper subset of P.

For a poset P, and a natural numbers k and m, let $w_k(P)$ be the maximum cardinality of a subposet Y of P of height at most k, and let $h_m(P)$ be the maximum cardinality of a subposet Z of P of width at most m. By convention, we set $w_0(P) = 0 = h_0(P)$. Clearly, $w_1(P) = w(P)$, the ordinary

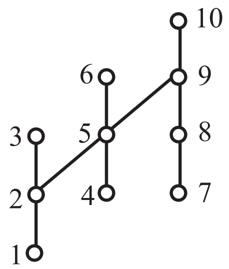

Fig. 2.4. Maximum Size Chain and Antichain Families

width of P, while $h_1(P) = h(P)$, the ordinary height of P. Furthermore, $w_k(P) = |P|$ when $k \geq h(P)$ and $h_m(P) = |P|$ when $m \geq w(P)$. When $k \leq |P|$, $w_k(P)$ is the maximum size of a k-family of antichains in P, while $h_m(P)$ is the maximum size of an m-family of chains of P when $m \leq |P|$.

Example 2.6.1 *For the poset P shown in Figure 2.6.1, $w_1(P) = 3$; $w_2(P) = 6$; $w_3(P) = 8$; $w_4(P) = 9$ and $w_k(P) = 10$ for all $k \geq 5$. Also, $h_1(P) = 4$, $h_2(P) = 8$ and $h_m(P) = 10$ for all $m \geq 3$.*

2.6.1 Bounding the Size of Families

When $\mathcal{S}$ is a family in a poset P and r is a natural number, we define the quantity $v_s(\mathcal{S})$ by:

$$v_r(\mathcal{S}) = r|\mathcal{C}| + |P - \langle\mathcal{C}\rangle|$$

Proposition 2.6.2 *Let m and k be natural numbers and let P be a poset on n points. Also, let $\mathcal{C}$ be an family of chains and $\mathcal{A}$ a k-family of antichains in P. Then*

$$|\langle\mathcal{A}\rangle| + |\langle\mathcal{C}\rangle| \leq km + n.$$

Proof We establish the equivalent statement:

$$|\langle\mathcal{A}\rangle| \leq km + \big(n - \langle\mathcal{C}\rangle|\big).$$

To see this, note that $\langle\mathcal{A}\rangle$ has height at most k, and any subset of P having height at most k contains at most k elements from each of the chains in $\mathcal{C}$, together with some or all of the elements not in $\langle\mathcal{C}\rangle$. $\square$

Proposition 2.6.3 *Let m and k be natural numbers and let P be a poset on n points. Also, let $\mathcal{C}$ be an family of chains and $\mathcal{A}$ a k-family of antichains in P. Then*

$$w_k(P) \leq v_k(\mathcal{A}) \quad \text{and} \quad h_m(P) \leq v_m(\mathcal{C}).$$

And in particular

$$w_k(P) + h_m(P) \leq km + n.$$

Definition. Let m and k be natural numbers and let P be a poset on n points. Also, let $\mathcal{C}$ be an family of chains and $\mathcal{A}$ a k-family of antichains in P. We say that $\mathcal{C}$ is *k-full* if $w_k(P) = v_k(\mathcal{C})$ and we say that $\mathcal{A}$ is *m-full* if $h_m(P) = v_m(\mathcal{A})$.

Example 2.6.4 *Returning to the* 10*-element poset shown in Figure 2.6.1, let*

$$\mathcal{C}_1 = \big\{\{1,2,3\},\{4,5,6\},\{7,8,9,10\}\big\} \quad \text{and} \quad \mathcal{C}_2 = \big\{\{1,2,5,9,10\},\{7,8\},\{4,6\}\big\}.$$

Then $v_k(\mathcal{C}_1) = 3k$ for $k = 1,2,3$ and $v_k(\mathcal{C}_1) = 10$ for all $k \geq 4$. It follows that $\mathcal{C}_1$ is k-full for all $k \geq 1$ except $k = 3$.

On the other hand, $v_1(\mathcal{C}_2) = 4$, $v_2(\mathcal{C}_2) = 7$, $v_3(\mathcal{C}_3) = 8$, $v_4(\mathcal{C}_2) = 9$ and $v_5(\mathcal{C}_2) = 10$ for all $m \geq 5$. It follows that $\mathcal{C}_2$ is k-full if and only if $k \geq 3$.

Now let

$$\mathcal{A} = \big\{\{2,4,7\},\{3,5,8\},\{6,9\}\big\}$$

Then $v_1(\mathcal{A}) = 5$, $v_2(\mathcal{A}) = 8$ and $v_m(\mathcal{A}) = 10$ for all $m \geq 3$. So $\mathcal{A}$ is m-full for all $m \geq 1$.

2.6.2 Statement of the Theorems

With this background discussion in mind, here is the full statement—in aggregate form—of the theorems of Greene [99] and Greene-Kleitman [99]. The first part is actually the theorem due to Greene and Kleitman, while the second part is due to Greene.

Theorem 2.6.5 (Greene-Kleitman Theorem) (**The Theorems of Greene and Kleitman**) *Let k and m be natural numbers and let P be a poset. Then*
There is a family $\mathcal{C}$ of chains in P which is both k-full and $(k+1)$-full.
There is a family $\mathcal{A}$ of antichains in P which is both m-full and $(m+1)$-full.

Example 2.6.1 only hints at the challenges to finding families of chains that are k-full for multiple values of k. The full extent of the difficulties is revealed in the following theorem of West [99].

Theorem 2.6.6 *Let h and w be integers with $h, w \geq 4$. Then there exist poset P_h and Q_w so that:*

(i) *For every pair k_1 and $k-2$ with $1 \leq k_1, k_2 < h$ and $|k_1 - k_2| \geq 2$, there is no family of chains in P_h which is both k_1-full and k_2-full.*
(ii) *For every pair m_1 and $m-2$ with $1 \leq m_1, m_2 < w$ and $|m_1 - m_2| \geq 2$, there is no family of antichains in Q_w which is both k_1-full and k_2-full.*

Chapell [99] has constructed even smaller examples than the posets used by West in proving Theorem **??**, and the poset shown in Figure 2.6.1 is one of his examples. Further details on Chappell's examples are included in the exercises.

One of the exercises at the end of the chapter is to construct the family $\{Q_w : w \geq 4\}$ as complements of the posets shown in Figure 11.

2.6.3 Saturated Chain and Antichain Partitions

Before proceeding with the proof, we pause to comment that many authors present the theorems of Greene and Kleitman Theorem in an alternative form, using chain and antichain partitions. We pause briefly here to show that the two approaches are entirely equivalent.

Let $\mathcal{H}$ be a chain partition of the poset P and define the quantity $e_k(\mathcal{H})$ by setting

$$e_k(\mathcal{H}) = \sum_{C \in \mathcal{H}} \min\{k, |C|\}.$$

Clearly $w_k(\mathcal{H}) \leq e_k(\mathcal{H})$ for every chain partition $\mathcal{H}$. A chain partition $\mathcal{H}$ is said to be *k-saturated* if $w_k(P) = e_k(\mathcal{H})$. The chain portion of Theorem 2.6.8 can then be restated as follows:

Theorem 2.6.7 *For every poset P and every positive integer k, there is a chain partition of P which is simultaneously k and $k+1$-saturated.*

To see that the two statements are equivalent, we note that if we have a family $\mathcal{C}$ of chains in P that is k-full, we can extend $\mathcal{C}$ to a chain partition $\mathcal{H}$ by adding all elements of $P - \langle \mathcal{C} \rangle$ as single-element chains. It is easy to see that $\mathcal{H}$ is k-saturated. Conversely, if $\mathcal{H}$ is a k-saturated chain partition of P, then we can remove all chains having k or fewer elements to obtain a k-full family $\mathcal{C}$. Similar remarks apply for the antichain portion of Theorem 2.6.8.

2.6.4 Preliminary Steps

Despite our insistence on maintaining full duality in statments and presentation for chains and antichains, we will have to tilt in one direction or the other to move forward with the proof. For reasons that will become clear, we focus on finding families of chains and our goal becomes to prove the following statement, which as we have already noted, is known as the Greene-Kleitman theorem:

Theorem 2.6.8 *For every non-negative integer k and every finite poset P, there is a chain family $\mathcal{C}$ in P that is simultaneously k-full and $(k+1)$-full.*

Our next step is a lemma due to Saks [99].

Lemma 2.6.9 *If k is a positive integer and P is a finite poset, then*

$$w_k(P) = w(P \times \mathbf{k}).$$

Proof Let $w_k(P) = t$ and let Y be a t-element subset of P having height at most k. Partition Y into k antichains $A_1, \ldots, A_k$ as in the proof of Theorem 2.2, i.e., A_1 is the set of minimal elements of Y and A_{i+1} is the set of minimal elements in the subposet obtained from Y after $A_1, A_2, \ldots, A_i$ have been removed. The key feature of this partition is that if $1 \leq i < j \leq h$, $x \in A_i$ and $y \in A_j$, then $x \not> y$ in P.

For $i = 1, \ldots, k$, set $B_i = \{(a, k-i) : a \in A_i\}$ and $B = B_1 \cup B_2 \cup \cdots \cup B_k$. It is clear that B is an antichain in $P \times \mathbf{k}$ and $|B| = |Y|$. Thus $w(P \times \mathbf{k}) \geq w_k(P)$.

Conversely, if $w(P \times \mathbf{k}) = s$ and B is an s-element antichain in $P \times \mathbf{k}$, set $B_i = \{b : (b, i) \in B\}$, for each $i = 0, \ldots, k-1$ and then set $Z = B_0 \cup B_1 \cup \cdots \cup B_{k-1}$. Note that the sets in $\{B_0, B_1, \ldots, B_{k-1}\}$ are pairwise disjoint, so Z is a subposet of P having cardinality s and height at most k. This shows $w_k(P) \geq w(P \times \mathbf{k})$. □

2.6.5 Details of the Proof

Our basic line of attack for proving the Greene-Kleitman theorem should now be apparent. First, we assume that $k+1 \leq h$, otherwise the claim holds trivially. In this case, we set $m = w_{k+1}(P) - w_k(P)$ and note that

$m > 0$. We then consider a chain partition $\mathcal{F}$ of $P \times \mathbf{k+1}$, and make a series of small changes while continuing to refer to the chain partition with the symbol $\mathcal{F}$. Throughout, we let $M_i(\mathcal{F}) = \{x \in P : (x, i)$ is the top point of some chain in $\mathcal{F}\}$. Again, the meaning of these sets changes over time.

We will use the same terminology that we used earlier in proofs of Dilworth's theorem. Specifically, we say that x covers y in $\mathcal{F}$ when there is a chain C in $\mathcal{F}$ so that x occurs immediately above y in C. In particular, we do not require that x cover y in P when x covers y in $\mathcal{F}$.

The goal for the series of alterations is to transform $\mathcal{F}$ into a family satisfying the following four properties:

$GK(1)$. $|\mathcal{F}| = w_{k+1}(P)$.
$GK(2)$. $|M_0(\mathcal{F})| = w_{k+1}(p) - w_k(P)$.
$GK(3)$. $M_{k-1}(\mathcal{F}) \subseteq M_{k-2}(\mathcal{F}) \subseteq M_1(\mathcal{F}) \subseteq M_0(\mathcal{F})$.
$GK(4)$. If $x \in M_k(\mathcal{F}) - M_{k-1}(\mathcal{F})$, then (x, k) covers $(x, k-1)$ in $\mathcal{F}$.

Before proceeding with the details of the argument, let's skip to the end and see why these four properties are key. Assuming all four hold, consider the chain partition of P induced by the restriction of the chains in $\mathcal{F}$ to level k. Then let $\mathcal{C}$ denote the non-trivial chains in this family. Then all of the top elements of chains in $\mathcal{C}$ belong to $M_k(\mathcal{F}) \cap M_{k-1}(P)$. Let $m = |\mathcal{C}|$ and let $t = |P - \langle \mathcal{C} \rangle|$. Note that $t \geq |M_k(\mathcal{F}) - M_{k-1}(P)$. In view of $GK(2)$ and $GK(3)$, we see that

$$\begin{aligned} w_{k+1}(P) - w_k(P) &= |M_0(\mathcal{F})| \\ &\geq |M_1(\mathcal{F})| \geq \cdots \geq |M_{k-1}(\mathcal{F})| \\ &\geq |M_k(\mathcal{F}) \cap M_{k-1}(P)| \\ &\geq m \end{aligned}$$

Also

$$\begin{aligned} w_{k+1}(P) &= |\mathcal{F}| \\ &= t + |M_0(\mathcal{F})| + |M_1(\mathcal{F})| + \cdots + |M_{k-1}(\mathcal{F})| + |M_k(\mathcal{F}) \cap M_{k-1}(\mathcal{F})| \\ &\geq (k+1)m + t \end{aligned}$$

On the other hand, we know that $w_{k+1}(P) \leq (k+1)m + t$ from Proposition 2.6.3. This shows that $w_{k+1}(P) = (k+1)m + t = v_{k+1}(\mathcal{C})$, so that $\mathcal{C}$ is $(k+1)$-full. It also shows that

$$M_0(\mathcal{F}) = M_1(\mathcal{F}) = \cdots = M_{k-1}(\mathcal{F}) = M_k(\mathcal{F}) \cap M_{k-1}(P)$$

and that each of these sets has exactly m elements. Since $w_{k+1}(P) - w_k(P) = m$, it follows that $w_k(P) = km + t = v_k(\mathcal{C})$ so that $\mathcal{C}$ is also k-full.

Now on to the construction of a family $\mathcal{F}$ satisfying $GK(1)$ through $GK(4)$. We start the construction by applying Corollary 2.4.8 to $P \times \mathbf{k+1}$ with the upset $U = \{(x,i) : x \in P, 1 \leq i \leq k\}$. We note that U is isomorphic to $P \times \mathbf{k}$ so the width of U is $w_k(P)$. It follows that $P \times \mathbf{k+1}$ has a chain partition $\mathcal{F}$ properties $GK(1)$ and $GK(2)$.

Before we tackle $GK(3)$ and $GK(4)$, we need an additional property† which is not listed as $GK(5)$ since it is not essential to the closing argument.

Local Cover Property. If (x,i) covers (y,j) in $\mathcal{F}$, then either (1) $i = j$ or (2) $x = y$ and $i = j + 1$.

To achieve this property, we make alterations called *insertions.* Whenever there is a chain C in $\mathcal{F}$ with (x,i) covering (y,j) in C, but neither condition (1) nor (2) holds, we find the chain C' in $\mathcal{F}$ containing the point $(y, j+1)$, remove it from C' and insert it between (x,i) and (y,j) in C. Observe that performing an insertion does not change the fact that $\mathcal{F}$ satisfies properties $GK(1)$ and $GK(2)$. To see that the sequence of insertions must end, consider the vector $(a_0, a_1, \ldots, a_{k-1})$ where a_j counts the number of points y from P for which (y,j) is covered by the point $(y, j+1)$ in $\mathcal{F}$. When the insertion described above is performed, this vector advances lexicographically, since (y,j) is now covered by $(y, j+1)$ and all other covers contributing to the counts in $(a_0, a_1, \ldots, a_j)$ are unaffected.

Next, we aim for $GK(3)$ and $GK(4)$. Here the alterations are called *switches.* In fact, we describe two different types of switches.

Type 1. Suppose there is some integer i with $1 \leq i \leq k$ for which $x \in M_i(\mathcal{F}) - M_{i-1}(\mathcal{F})$ and (x,i) does not cover $(x, i-1)$ in $\mathcal{F}$. Let C be the chain in $\mathcal{F}$ containing (x,i) as its top element. Since $(x, i-1) \notin M_{i-1}(\mathcal{F})$, there is a unique point y for which $(y, i-1)$ covers $(x, i-1)$ in $\mathcal{F}$. Then let C' be the chain containing the point (y,i). Note that (y,i) cannot by the lowest point of C', for this would imply that $C \cup C'$ is a chain which would result in a chain partition of $P \times \mathbf{k+1}$ of size $w_{k+1}(P) - 1$. However, if we let C'' be the portion of C' consisting of (y,i) and all the points of C' which are above (y,i), then $C \cup C''$ is a chain. We then modify $\mathcal{F}$ by removing C and C' from $\mathcal{F}$ and replacing them by $C \cup C''$ and $C' - C''$.

Type 2. Suppose that there is some i with $1 \leq i \leq k-1$ for which $x \in M_i(\mathcal{F}) - M_{i-1}(\mathcal{F})$ and (x,i) covers $(x, i-1)$ in $\mathcal{F}$. Let C be the chain

† Although we elect not to take this course here, the reader may note that the Local Cover Property can be assured just by paying attention to details in the application of the Gallai-Millgram theorem.

for which (x, i) is the top element. Then let C' be the chain containing $(x, i+1)$. Again, we note that $(x, i+1)$ is not the least element of C', so we may assume that $(y, i+1)$ is the element covered by $(x, i+1)$. Now let C'' consist of $(x, i+1)$ and all points of C' which are larger than $(x, i+1)$, then we modify $\mathcal{F}$ by first removing C and C' and replacing them by $C \cup C''$ and $C' - C''$.

It is easy to see that the two types of switches preserve properties $GK(1)$, $GK(2)$ and the local cover property. Also, they *seem* to be steering us towards a family satisfying $GK(3)$ and $GK(4)$ as well. The only remaining hurdle is to show that we will eventually reach a stage where no more switches are available. To accomplish this, we let $(b_0, b_1, b_2, \ldots, b_{k-1})$ be the vector where b_i counts the number of distinct points x from P so that if (x, i) is covered by (y, j) in $\mathcal{F}$, then $(x, i+1)$ is also covered by $(y, j+1)$. Note that this definition does not require x and y to be distinct. Regardless, each time we carry out a switch, this vector increases lexicographically. It follows then that we will eventually reach a partition $\mathcal{F}$ satisfying $GK(1)$ through $GK(4)$.

With this observation, our proof of the Greene-Kleitman theorem is complete.

2.6.6 Co-ordinated Chain Partitions

Let P be a poset and let k be a positive integer. With a chain family $\mathcal{C}$ in P, we associate a special kind of chain partition $\mathcal{G}$ of $P \times \mathbf{k}$. The partition $\mathcal{G}$ is called a *co-ordinated chain partition* of $P \times \mathbf{k}$ and the family $\mathcal{C}$ is called the *template* for $\mathcal{G}$. If $\mathcal{C}$ is an m-family, the partition $\mathcal{G}$ has km chains of the form: $\{C \times \{i\} : C \in \mathcal{C}, 0 \leq i < k\}$. These chains are called *horizontal* chains. For each $x \in P - \langle\mathcal{C}\rangle$, $\mathcal{G}$ contains a k-element chain $\{(x, i) : 0 \leq i < t\}$. These chains are called *vertical* chains. So altogether, $\mathcal{G}$ contains $v_k(\mathcal{C})$ chains.

Now let us return to the proof of the claim for chain families which we concluded just above. Again, we identified a family $\mathcal{C}$ of non-trivial chains of P by considering the restriction of the chains in $\mathcal{F}$ to level k. Now let $\mathcal{G}$ be the co-ordinated chain partition of $P \times \mathbf{k+1}$ having $\mathcal{C}$ as its template. Then $\mathcal{G}$ also satisfies $GK(1)$ through $GK(4)$.

We note here one immediate consequence of the Greene-Kleitman theorem, which seems surprisingly hard to prove from first principles.

Corollary 2.6.10 *For any poset P and any k,*

$$w_{k+2}(P) - w_{k+1}(P) \leq w_{k+1}(P) - w_k(P).$$

Proof Let $\mathcal{G}$ be the co-ordinated chain partition having $\mathcal{C}$ as template. Recall that $\mathcal{C}$ has $m = w_{k+1}(P) - w_k(P)$ non-trivial chains, each having size at least $k+1$. Then let $\mathbf{H}$ be the co-ordinated chain partition of $P \times \mathbf{k+2}$ having $\mathcal{C}$ as template. Evidently,

$$w_{k+2}(P) = w(P \times (\mathbf{k+2})) \leq |\mathcal{H}| = m + |\mathcal{G}| = 2w_{k+1}(P) - w_k(P).$$

The stated inequality then follows immediately. $\square$

2.7 Towards Greene's Theorem

In this section, we prepare to prove the following theorem, known as Greene's theorem.

Theorem 2.7.1 *For every non-negative integer m and every finite poset P, there is an antichain family $\mathcal{A}$ in P that is simultaneously m-full and $(m+1)$-full.*

Corollary 2.6.10 was an immediate consequence of our proof of the Greene-Kleitman theorem. By way of contrast, we will establish the dual form of this corollary first and then derive Greene's theorem as a corollary.

2.7.1 Orthogonal Families

Let m and k be natural numbers and let P be a poset on n points. Then let $\mathcal{C}$ be an m-family of chains, and let $\mathcal{A}$ be a k-family of antichains. If

$$|\langle \mathcal{C} \rangle| + |\langle \mathcal{A} \rangle| = n + km,$$

we say that $\mathcal{C}$ and $\mathcal{A}$ are *orthogonal*. When $\mathcal{C}$ and $\mathcal{A}$ are orthogonal, we note that:

- Each chain in $\mathcal{C}$ intersects each antichain in $\mathcal{A}$;
- Each element of P is in $\langle \mathcal{C} \rangle \cup \langle \mathcal{A} \rangle$;
- $\mathcal{C}$ is k-full, and $\mathcal{A}$ is m-full,
- $|\langle \mathcal{C} \rangle| = h_m(P) = n + km - w_k(P)$,
- $|\langle \mathcal{A} \rangle| = w_k(P) = n + km - h_m(P)$,
- Every maximum size m-family of chains is orthogonal to every maximum size k-family of antichains.

For an n-element poset P, we define the set $\Gamma(P)$ to be the set of ordered pairs (k, m) such that $h_m(P) + w_k(P) = n + km$. As we have seen, there are many different equivalent characterizations of $\Gamma(P)$. For instance, $(k, m) \in \Gamma(P)$ if and only if every maximum size m-family $\mathcal{C}$ of chains is k-full.

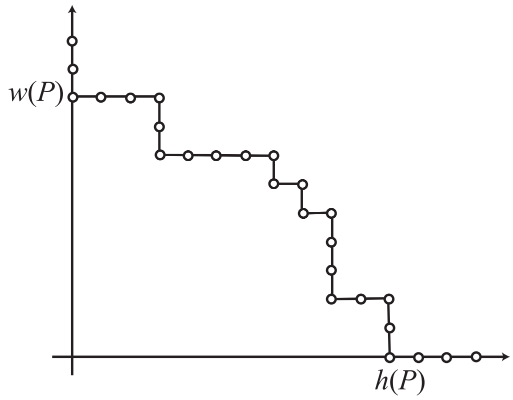

Fig. 2.5. The structure of the set $\Gamma(P)$.

So the Greene-Kleitman Theorem says that, for every k, there is some m such that both (k, m) and $(k+1, m)$ are in $\Gamma(P)$. Furthermore, as we have already noted, the (unique) value of m is $w_{k+1}(P) - w_k(P)$.

Our aim is to show that, for any poset P with $|P| = n$, $\Gamma(P)$ has a structure of the form illustrated in Figure 2.7.1. For a start, note that $(0, m) \in \Gamma(P)$ if and only if $h_m(P) = n$, i.e., if and only if $m \geq w(P)$. Similarly, $(k, 0) \in \Gamma$ if and only if $k \geq h(P)$. We also know that $(1, w(P))$ and $(h(P), 1)$ are in $\Gamma(P)$. So what we want to establish is that the remaining elements of $\Gamma(P)$ form a "lattice path" from $(1, w(P))$ to $(h(P), 1)$, with each step being either down or to the right.

To formalize this notion, we make the following definition. For each m, let $\Gamma_m(P) = \{k : (k, m) \in \Gamma(P)\}$. Our aim is then to prove the following two claims.

Claim 1. If m is a natural number, then $\Gamma_m(P)$ is a set of consecutive integers.

Proof. Suppose that k and $k+s$ are, respectively, the least and greatest members of $\Gamma_m(P)$. We show that all the integers between them also belong to $\Gamma_m(P)$. First, we have the following equations for k and $k+s$.

$$w_{k+s}(P) = n + (k+s)m - h_m(P) \quad \text{and} \quad w_k(P) = n + km - h_m(P).$$

Thus $w_{k+s}(P) - w_k(P) = sm$.

But from Corollary 2.6.10, we know that

$$w_{k+s}(P) - w_k(P) = \sum_{i=1}^{s} w_{k+i}(P) - w_{k+i-1}(P) \leq s\big(w_{k+1}(P) - w_1(P)\big)$$

Moreover, from Proposition 2.6.3, we know that

$$w_{k+1}(P) \leq n + (k+1)m - h_m(P) = w_k(P) + m,$$

so that $w_{k+1}(P) - w_k(P) \leq m$. It follows that $w_{k+i}(P) - w_{k+i-1}(P) = m$, for each $i = 1, 2, \ldots, s$. Therefore $w_{k+i}(P) = (k+i)m - h_m(P)$ for each $i = 0, 1, 2, \ldots, s$, which means that $k + i \in \Gamma_m(P)$ for each $i = 1, 2, \ldots, s-1$. This completes the proof of the claim.

Claim 2. Let k be a non-negative number. Then let m and r be, respectively, the least and greatest natural numbers so that $k \in \Gamma_m(P)$ and $k \in \Gamma_r$. Then $k \in \Gamma_p(P)$ for every p with $m < p < r$.

Proof. If $r \leq m+1$, there is nothing to prove, so we assume $r \geq m+2$. The hypothesis requires that

$$h_m(P) = n + km - w_k(P) \quad \text{ and } h_r(P) = n + kr - w_k(P).$$

so that $h_r(P) - h_m(P) = (r-m)k$. Furthermore, we know that

$$h_{m+1}(P) \leq n + k(m+1) - w_k(P) = h_m(P) + k$$

so that $h_{m+1}(P) - h_m(P) \leq k$. So the conclusion of this second claim follows just as in the proof of the first claim—if we can establish the following lemma.

Lemma 2.7.2 *Let P be a poset and let m be a natural number. Then*

$$h_{m+1}(P) - h_m(P) \geq h_{m+2}(P) - h_{m+1}(P).$$

Proof We apply techniques from network flows. We set up a directed graph D with vertex set $P \cup \{u, v\}$, where S (the *source*) and T (the *sink*) are not in P. We have an arc from S to every vertex of P, and from every vertex of P to T. Also, we have an arc (x, y) from x to y whenever $x < y$ in P.

Let $\mathcal{C}$ be an m-family of chains with $|\langle \mathcal{C} \rangle| = h_m(P)$. The family $\mathcal{C}$ corresponds in a natural way to a set of m internally vertex-disjoint paths from S to T in D: a chain $x_1 < x_2 < \cdots < x_r$ corresponds to the directed path $(S, x_1, x_2, \ldots, x_r, T)$ in D. Let A be the set of arcs on these paths, so $|A| = h_m(P) + m$. Similarly, if $\mathcal{C}'$ is an $(m+2)$-family of chains with $|\langle \mathcal{C}' \rangle| = h_{m+2}(P)$, then it corresponds to a set of $m+2$ paths with arc-set A' of size $h_{m+2}(P) + m + 2$.

The basic idea is that "$A' - A$" consists of two $S \to T$ paths, and adding the longer one to A results in a large $(m+1)$-family of chains. The formal justification takes some time, however.

Consider an auxiliary digraph D^* with vertex set $P \cup \{S, T\}$ and arcs of

Fig. 2.6. Linking arcs in D^*.

two types: firstly, every arc of $A' - A$ appears in D^*, and secondly, for every arc (x, y) of $A - A'$, the arc (y, x) appears in D^*. We observe that, for every vertex x in P, the in-degree of x in D^* is equal to the out-degree, while the out-degree of S exceeds its in-degree by 2, and the in-degree of T exceeds its out-degree by 2. In total, there are $h_{m+2}(P) - h_m(P) + 2$ more arcs of the first type than of the second type.

We wish to link all the arcs of D^* into walks and circuits. For a vertex x of D^* of in-degree and out-degree 1, we simply link the arc entering x with the arc leaving x. The digraph D^* may contain vertices $x \in X$ of out-degree and in-degree 2. This is the case only when x is on a chain $\ldots y < x < z \ldots$ in $\mathcal{C}$, and on a chain $\ldots y' < x < z' \ldots$ in $\mathcal{C}'$, with $y \neq y'$ and $z \neq z'$. See Figure 2.7.1. In this case, we link the arc (y', x) of D^* with the arc (x, y), and link the arc (z, x) with the arc (x, z'). This means that, if there is an arc (y, x) in A, and a different arc (y', x) in A', then $(y'x)$ is linked to (x, y) in D^*. Finally, if there are any arcs entering S, we link them in an arbitrary way to arcs leaving T, and similarly for arcs leaving v.

In this way, we decompose the arc-set of D^* into two $S \to T$ walks, and possibly some circuits. The aim is to "augment" the arc set A by one of these walks or circuits, resulting in another family of chains in P.

First, suppose one of the circuits H has more arcs of the first type than of the second type. Then alter A by adding all the arcs of H of the first type (which are in A' and not A) and, for each arc (y, x) of H of the second type, removing (x, y). Let A^* denote the new arc-set. Evidently all net in-degrees are the same in A^* as in A, and there are more arcs in A^* than in A.

Can a vertex x of X have in-degree 2 in A^*? This would mean that an arc (y, x) is present in A, and another arc (y', x) is an arc of H of the first type: but in this case the arc (x, y) would be an arc of the second type in D^*, linked to (y', x), so in H, and not in A^*. So every node of X has in-degree and out-degree at most one in A^*. Thus A^* corresponds to an m-family of chains in P, with more elements than $\mathcal{C}$, a contradiction.

Thus one of the two $S \to T$ walks W has at least $\frac{1}{2}\left(h_{m+2}(P) - h_m(P)\right)+1$ more arcs of the first type than of the second type. Just as before, we can augment A by adding in the arcs of W of the first type, and removing the reverses of the arcs of the second type. Again we get an arc-set corresponding to a family of chains, but now there are $m+1$ chains, as the out-degree of S has been increased. The total number of vertices on these chains is at least $\frac{1}{2}\left(h_{m+2}(P) + h_m(P)\right)$, so $h_{m+1}(P)$ is at least this large.

This completes the proof of the lemma. □

As we noted before, with the Lemma in hand, Claim 2 is valid and in turn, the proof of Greene's theorem is complete.

2.7.2 Reformulating Greene's Theorem and Accompanying Machinery

There are various attractive ways of re-formulating Theorem **??**. Here, for instance, is Frank's version [**?**].

Corollary 2.7.3 *For every poset P, there is a sequence $\mathcal{B}_0, \mathcal{B}_1, \ldots, \mathcal{B}_m$ where:*

- *each $\mathcal{B}_i$ is either a family of chains or of antichains,*
- *$\mathcal{B}_0$ is a chain-partition of P, and $\mathcal{B}_1$ consists of one antichain,*
- *if $\mathcal{B}_i$ $(i \geq 2)$ is a family of chains, then it contains one fewer chain than the previous family of chains on the list, while if it is a family of antichains, it contains one more antichain than the previous family of antichains,*
- *the final family of chains on the list contains just a single chain, and the final family of antichains is an antichain-partition of P, and*
- *each $\mathcal{B}_i$ $(i \geq 1)$ is orthogonal to the previous member of opposite type on the list.*

Fig. 2.7. A graph with no family of independent sets orthogonal to a 2-family of cliques

Proof The list of families is obtained by walking along $\Gamma(P)$ from $(0, w(P))$ to $(h(P), 1)$; on stepping from $(k-1, m)$ to (k, m), put any maximum-size k-family of antichains into the list, and stepping from $(k, m-1)$ to (k, m), put any maximum-size m-family of chains in the list. □

Another reformulation of Theorem **??** is in terms of partitions of the integer n, the size of P: see Exercise 15.

2.7.3 Greene-Kleitman and Perfect Graphs

The reader may well be asking whether this is all part of a much more general theory, and whether results similar to the theorems of Greene and Kleitman hold for perfect graphs in general. What would this mean? Given a (perfect) graph G, let $c_k(G)$ be a maximum-size family of k disjoint cliques in G. Can we always find, for some m, an "orthogonal" family of m disjoint independent sets of size $n+km-c_k(G)$? Unfortunately, even this is not true, as shown by the example in Figure 2.7.3, first considered in this context by Greene [**?**], with $k = 2$.

This graph, or sometimes its complement, can be used to refute most other possible generalizations of the results of the last two sections to perfect graphs.

2.8 Erdős-Szekeres revisited

The theorems of Greene and Kleitman presented in the previous section are considerably more powerful than Dilworth's theorem and its dual. So it is not unreasonable to expect that applying these results to the setting of the result of Erdős and Szekeres, Theorem **??**, will prove fruitful. Indeed, we will now obtain what looks like a considerable extension of that result, with very little extra work.

Theorem 2.8.1 *For every n-element poset P, there is some positive integer m, at most $\sqrt{n}$, such that either there are m chains of P covering at least $\lceil \frac{1}{2}(n+m^2) \rceil$ elements, or there are m antichains of P covering at least $\lceil \frac{1}{2}(n+m^2) \rceil$ elements.*

Proof By Theorem **??** there is some m for which $(m, m) \in \Gamma$. This implies that there is an m-family $\mathcal{C}$ of chains, and an m-family $\mathcal{A}$ of antichains, that are orthogonal. Then we have $|\langle \mathcal{C} \rangle| + |\langle \mathcal{A} \rangle| = n + m^2$, so at least one of the two families has total size at least $\lceil \frac{1}{2}(n+m^2) \rceil$, as claimed. □

Suppose that, for some m, there are m chains covering $\frac{1}{2}(n+m^2)$ elements of the poset P. Then the average length of these chains is at least

$$\frac{1}{2}\left(\frac{n}{m} + m\right) = \sqrt{n} + \frac{(\sqrt{n} - m)^2}{2m} \geq \sqrt{n}.$$

Corollary **??** guarantees us just one chain or antichain of size at least $\lceil \sqrt{n} \rceil$; Theorem **??** tells us that either we can find one much larger than that, or we can find a large collection of chains or antichains of average size at least $\sqrt{n}$.

A somewhat weaker but more striking result is the following, stating that we can cover more than half of the elements of P using either chains or antichains, with the average size in each case being at least $\sqrt{n}$.

2.9 Further Applications of Greene-Kleitman

Lemma 2.9.1 *Let k and m be positive integers. Let P be an n-element poset of height at least k and width at least m, and let a, b, c and d be the integers such that $h_m(P) = am + b$ $(0 \leq b < m)$ and $w_k(P) = ck + d$ $(0 \leq d < k)$. Then*

$$n \leq \max\{ac + b + d, h_m(P) + w_k(P) - k - m + 1\}.$$

Proof By the Greene-Kleitman Theorem, there is some r such that $w_k(P) + h_r(P) = n + kr$. Since $k \leq h(P)$, we can take $r \geq 1$. This requires a k-full r-family of chains to intersect an r-full k-family of antichains in kr elements, so in particular is at most both $h_r(P)$ and $w_k(P) < (c+1)k$; the second inequality gives us that $r \leq c$. We distinguish two cases.

(1) If $r \geq m$, then $h_r(P) \leq ar + b$, since otherwise the longest m chains from an r-family of chains of size $h_r(P)$ will have more than $am + b$ members. Now we have $kr \leq h_r(P) \leq ar + b$, so $k \leq a$. Now we have

$$\begin{aligned} n &= w_k(P) + h_r(P) - kr \\ &\leq ck + d + ar + b - kr \\ &\leq ck + d + b + (a-k)c \\ &= ac + b + d, \end{aligned}$$

and we have the result in this case.

(2) Now suppose $r < m$. We have $h_r(P) + w_k(P) = n + kr$, and also, since the width of P is at least m, $h_m(P) - h_r(P) \geq m - r$. So

$$n \leq h_m(P) + w_k(P) - kr - m + r \leq h_m(P) + w_k(P) - k - m + 1,$$

since $r \geq 1$, and we are done in this case too.

□

This result is best possible. Indeed, given a, b, c, d, the first bound $(ac+b+d)$ is strictly larger than the second exactly when $(c-m)(a-k) > (k-1)(m-1)$, which requires $c > m$ and $a > k$ (if the reverse inequalities hold, one would need c or a equal to 0, contradicting the assumption that P has height at least k and width at least m). If these inequalities hold, take P to be the disjoint union of: b chains of height $a+1$, $c-b$ chains of height a, and one chain of height d. This poset P has $ac+b+d$ elements, $h_m(P) = am + b$, and $w_k(P) = ck + d$. As for the second bound, whatever values of h_m and w_k are specified, let P' be the disjoint union of a chain of height $h_m - m + 1 \geq 1$ and an antichain of size $w_k - k \geq 0$: then P' has the required values of h_m and w_k, and just $h_m(P') + w_k(P') - m - k + 1$ elements.

The fact that the first bound in Lemma 2.9.1 only applies in cases with $c \geq m$ and $a \geq k$ gives us the following more appealing but less precise version.

Corollary 2.9.2 *If P is an n-element poset of height at least k and width at least m, then*

$$n \leq \max\{\frac{h_m(P)w_k(P)}{mk}, h_m(P) + w_k(P) - k - m + 1\}.$$

The case $k = m = 1$ is just Corollary**??**. Since $h_m(P)/m$ and $w_k(P)/k$ are non-increasing functions of m and k respectively, while $hm(P) - m$ and $w_k(P) - k$ are non-decreasing, we get the best results when the two bounds are (almost) equal.

We now apply the lemma to get our extension of Corollary **??**, and hence of the theorem of Erdős and Szekeres.

Theorem 2.9.3 *Let s, t and p be integers with $p \le \min(\lfloor (s+1)/2 \rfloor, \lfloor (t+1)/2 \rfloor)$, and let P be any poset with $n \ge st + 2p - 1$ elements. Then either there are $\lfloor (s+1)/2 \rfloor$ chains covering at least $\lfloor (s+1)/2 \rfloor t + p$ elements of P, or there are $\lfloor (t+1)/2 \rfloor$ antichains covering at least $\lfloor (t+1)/2 \rfloor s + p$ elements of P.*

Proof If P is a counterexample, then

$$h_{\lfloor (s+1)/2 \rfloor} \le \left\lfloor \frac{s+1}{2} \right\rfloor t + p - 1$$
$$w_{\lfloor (t+1)/2 \rfloor} \le \left\lfloor \frac{t+1}{2} \right\rfloor s + p - 1$$

Thus we can apply Lemma 2.9.1 with $m = \lfloor (s+1)/2 \rfloor$, $k = \lfloor (t+1)/2 \rfloor$, $a = t$, $c = s$, $b = d = p - 1$. The result is trivial if P has height less than k or width less than m, as then we can cover the entire poset with either m chains or k antichains.

Now we have $ac + b + d = st + 2p - 2$, and

$$\begin{aligned} h_m(P) + w_k(P) - k - m + 1 &\le \frac{s+1}{2}(t-1) + \frac{t+1}{2}(s-1) + 2(p-1) + 1 \\ &= st + 2p - 2. \end{aligned}$$

Hence, by Lemma 2.9.1, we have $n \le st + 2p - 2$, a contradiction. □

The application to mononote subsequences of arbitrary sequences is exactly as in Theorem **??**. We state a slightly weaker, but perhaps more appealing, form of the result.

Theorem 2.9.4 *Let m and p be positive integers with $p \le \lfloor (m+1)/2 \rfloor$, and let P be a poset with $n \ge m^2 + 2p - 1$ elements. Then either there are $\lfloor (m+1)/2 \rfloor$ chains covering at least $\lfloor (m+1)/2 \rfloor m + p$ elements of P, or there are $\lfloor (m+1)/2 \rfloor$ antichains covering at least $\lfloor (m+1)/2 \rfloor m + p$ elements of P.*

Proof By Theorem 2.8.1, there is an $m \le m$ such that either there is an m-family of chains of size at least $\frac{1}{2}(n + m^2)$, or there is an m-family of antichains of this size. Without loss of generality, we assume the former.

For convenience, set $r = \lfloor (m+1)/2 \rfloor$. We consider two cases.

(1) If $m \geq r$, then it is enough to show that $\lceil \frac{1}{2}(n+m^2) \rceil \geq mm + p$, since then the largest r chains from the m-family will have total size at least $rm + p$, as required. But we have

$$\begin{aligned} \frac{n+m^2}{2} - mm - p &\geq \frac{m^2 + 2p - 1 + m^2 - 2mm - 2p}{2} \\ &= \frac{(m-m)^2 - 1}{2} \\ &\geq 0, \end{aligned}$$

except when $m = m$, in which case the required inequality holds since $\lceil \frac{1}{2}(n+m^2) \rceil \geq m^2 + p$.

(2) If $m < r$, then adding an appropriate number of singleton chains to the m-family gives us an r-family of chains of size at least $\lceil \frac{1}{2}(n+m^2) \rceil + r - m$. This is minimized when $m = 1$, when it is at least $\lceil \frac{m^2}{2} \rceil + r + p - 1 \geq rm + p$. (The final inequality can be checked separately in the cases of m even and m odd.)

□

Theorem 2.9.4 is best possible in a number of senses: (a) if the entire poset is the union of one chain and one antichain, each of size $m^2/2 + p$ (this corresponds to the case $m = 1$), then we cannot find a family of $r' = \lfloor (m+1)/2 \rfloor + 1$ chains or antichains of total size $r'm + p$; (b) if $n = m^2 + 2p - 2$, and P is made up of the union of $p - 1$ chains of height $m + 1$, $m - p + 1$ chains of height m, and one chain of height $p - 1$, then there is no union of p chains, or of p antichains, of total size $(m+1)p$. However, Theorem **??** still says more. One can also prove a version of Theorem 2.9.4 which is not symmetric between chains and antichains; we leave this to the exercises.

Just as in the beginning of this chapter, if we have a sequence of n distinct real numbers, we can form a poset on the set $[n]$ so that increasing subsequences of the sequence correspond to chains, while decreasing subsequences correspond to antichains. Theorem 2.9.4 then gives us the following extension of the theorem of Erdős and Szekeres.

Theorem 2.9.5 *In any sequence of $n = m^2 + 1$ distinct real numbers, either there are $\lfloor (m+1)/2 \rfloor$ disjoint increasing subsequences of average length greater than m, or there are $\lfloor (m+1)/2 \rfloor$ disjoint decreasing subsequences of average length greater than m.*

2.10 Families of Pairwise Disjoint Chains and Antichains

The closing sections of this chapter are concerned with some quite recent results which have dual statements for chains and antichains—much in the same spirit as the theorems of Greene and Kleitman. Also, the natural extremal posets showing that the results are best possible seem to have much in common. Although no direct link between the two topics is known, it would be quite interesting to determine if there is one.

In [99], Duffus and Sands initiated a study of the following problem: Fix an integer $k \geq 2$. Then find conditions that guarantee that a poset P has k pairwise disjoint maximal antichains. Just by considering the set of maximal points and the set of minimal points, they noted the following solution when $k = 2$.

Proposition 2.10.1 *A poset P has 2 pairwise disjoint maximal antichains if and only if no point of P is incomparable with all other points of P.*

But when $k \geq 3$, the problem becomes more subtle. However, Duffus and Sands we able to establish the following sufficient condition.

Theorem 2.10.2 (Duffus-Sands) *Let n and k be integers with $n \geq k \geq 3$, and let P be a finite poset. If $n \leq |C| \leq n + (n-k)/(k-2)$, for every maximal chain C in P, then P has k pairwise disjoint maximal antichains.*

For each pair n and k with $n \geq k \geq 3$, Duffus and Sands also constructed a poset $P(n,k)$ satisfying the following properties:

(i) If C is a maximal chain in $P(n,k)$, then $n \leq |C| \leq n + 1 + \lfloor (n-k)/(k-2) \rfloor$.

(ii) $P(n,k)$ does not have k pairwise disjoint maximal antichains.

These examples show that the inequality in Theorem 2.10.2 is best possible. We illustrate their construction with the diagram shown in Figure 2.10. Here the specific parameters are $n = 16$ and $k = 6$, but the modifications for other values of the two parameters should be clear.

In this same paper, Duffus and Sands also initiated the study of the dual problem for pairwise disjoint families of chains. Again the result for $k = 2$ is complete, although the argument (one of our exercises at the end of the chapter) is not entirely trivial.

Proposition 2.10.3 *A poset P has 2 pairwise disjoint maximal chains if and only if no point of P is comparable to all others.*

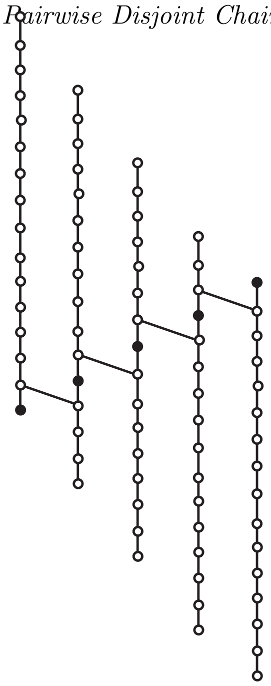

Fig. 2.8. Posets for Duffus-Sands Inequality

Duffus and Sands noted that the posets in their family $\{P(n, k) : n \geq k \geq 3\}$ have complements in which the role of comparability and incomparability are exchanged † Accordingly, it was natural for them to ask whether the dual version of their theorem might be true, noting that if the answer were positive, then again the result would be best possible.

Howard and Trotter [99] answered their question in the affirmative by proving the following result.

Theorem 2.10.4 (Howard-Trotter) *Let n and k be integers with $n \geq k \geq 3$, and let P be a finite poset. If $n \leq |A| \leq n + (n-k)/(k-2)$ for every maximal antichain A in P, then P has k pairwise disjoint maximal chains.*

Howard and Trotter actually proved a more technical and somewhat stronger result and derived Theorem 2.10.4 as a corollary. They then showed how a dual version of the technical result can be proved for families of pairwise disjoint maximal antichains, and as a consequence, provided an alternative proof of Theorem **??**. While the original Duffus-Sands proof of this

† In Chapter 9.99, we will learn that the posets in this family are 2-dimensional.

theorem remains of interest, we present here the approach taken by Howard and Trotter.

2.10.1 Cutsets and Support Structures

Let P be a finite poset. In discussions concerning families of pairwise disjoint maximal chains in P, we find it useful to apply well known concepts and techniques from network flows. In particular, we will employ the following basic proposition.

Proposition 2.10.5 *The maximum number of pairwise disjoint maximal chains in P equals the minimum cardinality of a set intersecting all maximal chains in P.*

In view of Proposition 2.10.5, the following notation and terminology becomes natural. We will say that a chain C in a finite poset P is *saturated* if either $|C| = 1$ or if $|C| = r > 1$ and $C = \{x_1 < x_2 < \cdots < x_r\}$, then x_i is covered by x_{i+1} for each $i = 1, 2, \ldots, r-1$.

A saturated chain in P whose least element is a minimal element of P will be called an *initial* chain. Dually a saturated chain whose greatest element is a maximal element of P will be called a *terminal* chain. A maximal chain is always saturated and is both an initial chain and a terminal chain. Trivially, for every point u in P, there is an initial chain whose greatest element is u, and there is a terminal chain whose least element is u. The union of these two chains is a maximal chain containing u.

Note also that whenever $u < v$ in P, there is always a saturated chain C with u the least element of C and v the greatest element of C. We say such a chain is a *linking* chain for u and v.

Let P be a finite poset and let W be a subset of P that intersects all maximal chains in P. We will refer to W as a *cutset* in P. Next, we will develop some additional structural information concerning cutsets.

Now let s be a positive integer, and let W be an s-element cutset in P. Then let r be the height of the subposet W, and let $W = W_1 \cup W_2 \cup \cdots \cup W_r$ be the canonical partition of W into antichains, i.e., W_i consists of the points in W whose height in W is i. Then for each $i = 1, 2, \ldots, r$, let A_i be the maximal elements of the set $\{x \in P : x \not> w, \text{ for all } w \in W_i\}$. It is obvious that A_i is a maximal antichain in P and that $W_i \subseteq A_i$. We refer to the maximal antichains in the family $\{A_i : 1 \leq i \leq r\}$ as *flat* antichains. Note that A_i and A_j need not be disjoint when $i \neq j$. However, the following important property does hold.

Claim 1. If $1 \leq i < j \leq r$, $u \in A_i$ and $v \in A_j$, then $u \not> v$ in P.

Proof. Suppose to the contrary that $u > v$ in P. Since v is a maximal element of the set $\{x \in P : x \not> w\}$, for all $w \in W_j\}$, then there exists some element $w \in W_j$ with $u > w$ in P. Since $i < j$, there is then some element $w' \in W_i$ with $w > w'$ in P. By transitivity, this implies that $u > w'$ in P with both u and w' belonging to the antichain A_i. The contradiction completes the proof of the claim.

Let u be an element of P. We say u is *reachable* if there is an initial chain C having u as its greatest element so that $C \cap W = \emptyset$. Evidently, no point of W is reachable. Also, all minimal elements of P that do not belong to W are reachable. On the other hand, no maximal element of P is reachable, as this would imply that there is a maximal chain in P that does not intersect W.

For each $i = 1, 2, \ldots, r$, let R_i denote the set of reachable points in the antichain A_i, and let $N_i = A_i - W_i - R_i$. Elements of N_i are not reachable. Here are two easy claims about reachable and non-reachable points. We include the proofs among the exercises at the end of the Chapter.

Claim 2. $N_1 = R_r = \emptyset$.

Claim 3. For each $i = 1, 2, \ldots, r-1$, R_i and N_{i+1} are disjoint sets and $R_i \cup N_{i+1}$ is an antichain in P.

We refer to the antichains in the family $\mathcal{S} = \{R_i \cup N_{i+1} : 1 \leq i \leq r-1\}$ as *slanted* antichains. Note that slanted antichains need not be maximal. Also, we refer to the family $\{A_i = W_i \cup R_i \cup N_i : 1 \leq i \leq r\}$ as the *support structure* for the cutset W in P. Strictly speaking, the support structure of a cutset W is determined entirely by W and P, but we find it useful to carry along the additional information given by the family of flat antichains, and the set of reachable elements.

Here is the technical result we are now positioned to prove.

Theorem 2.10.6 *Let P be a poset, let s denote the maximum number of pairwise disjoint maximal chains in P, and let W be an s-element cutset in P. If the height of W is r, the width of P is t and every maximal antichain in P has at least n elements, then*

$$rn \leq s + t(r-1) \tag{2.1}$$

Proof Let $\{A_i = W_i \cup R_i \cup N_i : 1 \le i \le r\}$ be the support structure of W. Since $|N_1| = |R_r| = 0$, it is immediate that

$$\sum_{i=1}^{r} |A_i| = s + \sum_{i=1}^{r} |R_i| + |N_i| = s + \sum_{i=1}^{r-1} |R_i \cup N_{i+1}|$$

Since $|A_i| \ge n$ for each $i = 1, 2, \ldots, r$ and $|R_i \cup N_{i+1}| \le t$, for each $i = 1, 2, \ldots, r-1$, inequality 2.1 follows. □

To see how our main Theorem now follows easily as a corollary to Theorem **??**, let n and k be integers with $n \ge k \ge 3$. Then let P be a finite poset in which every maximal antichain has at least n elements, and suppose that the width t of P is at most $n+(n-k)/(k-2)$. If P does not have k pairwise disjoint chains, then there is some positive integer s with $s < k$ for which there is an s-element cutset W in P. Let r denote the height of W. From Theorem **??**, we know that $rn \le s + t(r-1)$, and this inequality may be rewritten as $t \ge n + (n-s)/(r-1)$. Since $r \le s$ and $s \le k-1$, this implies

$$t \ge n + \frac{n-s}{r-1} \ge n + \frac{n-s}{s-1} \ge n + \frac{n-k+1}{k-2}.$$

This is a contradiction, since $t \le n + (n-k)/(k-2)$, and this remark completes the proof.

2.10.2 The Dual Problem

It is worth noting that the approach used to prove Theorems **??** and **??** cannot be applied (at least not without modification) to the original problem studied by Duffus and Sands. The reason is that the dual version of Proposition **??** is not valid. Specifically, it is not true that the maximum number of pairwise disjoint antichains in a finite poset P equals the minimum cardinality of a set intersecting all maximal antichains in P.

Here is a lemma whose proof is again deferred to the exercises.

Lemma 2.10.7 *For any integer $n \ge 2$ for which there exists a projective plane of order n, there exists a poset P_n in which the maximum number of pairwise disjoint antichains is 2, but the minimum cardinality of a set of points intersecting all maximal antichains is $2n$.*

In spite of the apparent difficulties presented by Lemma **??**, there is a natural framework within which we can derive a dual version of Theorem **??** and then proceed to derive the Duffus-Sands result as a corollary.

Let P be a finite poset and let t denote the height of P. Then, for each

$i = 1, 2, \ldots, t$, let $L_i = \max\{x : h_P(x) \le i\}$. We refer to $\{L_i : 1 \le i \le t\}$ as the family of *level* antichains in P. It is straightforward to verify that each level antichain is a maximal antichain. Furthermore, we have the following important property:

Proposition 2.10.8 *If $1 \le i < j \le r$, $u \in L_i$ and $v \in L_j$, then $u \not> v$ in P.*

The following key result admits an easy elegant proof.

Theorem 2.10.9 *The maximum number of pairwise disjoint level antichains is equal to the minimum number of points in a set intersecting all of them.*

Proof We show that that there is a partition $\{1, 2, \ldots, t\} = B_1 \cup B_2 \cup \cdots \cup B_s$, so that for each $p = 1, 2, \ldots, s$:

(i) $B_p = [b_p, c_p]$ is a block of consecutive integers with $b_p = 1 + c_{p-1}$ when $p > 1$.
(ii) There is a point x_p common to all antichains in $\{L_i : i \in B_p\}$.
(iii) If $c_p < i \le t$, then $L_i \cap L_{b_p} = \emptyset$.

Once this partition has been constructed, we will then have a family $\{L_{b_p} : 1 \le p \le s\}$ of s pairwise disjoint maximal antichains and an s-element set $W = \{x_p : 1 \le p \le s\}$ which intersects all level antichains.

The construction proceeds inductively Set $c_0 = 0$. Suppose for some $p \ge 1$, we have a value of c_{p-1} and if $p \ge 2$, the properties listed above hold for the blocks $B_1, B_2, \ldots, B_{p-1}$. If $c_{p-1} < t$, set $b_p = 1 + c_{p-1}$ and let c_p be the largest integer for which $c_p \le t$ and $L_{b_p} \cap L_{c_p} \ne \emptyset$. Then choose x_p as an element from $L_{b_p} \cap L_{c_p}$. It follows from Proposition 2.10.8 that x_p belongs to every antichain in $\{L_i : i \in B_p\}$. Furthermore, if $c_p < i \le t$, then $L_i \cap L_{b_p} = \emptyset$. □

Now we can state and prove a dual version for Theorem **??**

Theorem 2.10.10 *Let P be a poset, let s denote the maximum number of pairwise disjoint antichains in the family of level antichains in P, and let W be an s-element set intersecting all level antichains in P. Let r be the width of W and let $C_1, C_2, \ldots, C_r$ be maximal chains in P that cover W. If every maximal chain in P has at least n elements, then:*

$$rn \le s + t(r - 1). \tag{2.2}$$

Proof Let $x \in W$ and let $B = [b, c]$ be the set of consecutive integers from $\{1, 2, \ldots, t\}$ so that $x \in L_j$ if and only if $j \in B$. It follows that $h_P(x) = b$.

Furthermore, if $x \in C_i$, then there are no points in C_i that have height j where $b < j \leq c$. Since $|C_i| \geq n$ for each $i = 1, 2, \ldots, r$ and and we have eliminated points at all heights from $\{1, 2, \ldots, t\}$, except for the heights of elements of W, we conclude that $rn \leq rt - t + s$, which is equivalent to inequality 2.2. □

Note that Theorem 2.10.2 again follows immediately from this more technical result.

Exercises

2.1 What is the *maximum* value of $h(P)w(P)$, for an n-element poset P?

2.2 Let G be a graph. Suggest an expression for the width of the incidence poset of G. Prove your result.

2.3 Let $D[n] = ([n], |)$, where $|$ denotes strict divisibility. Find and justify formulae for the height and width of $D[n]$.

2.4 For each $m \in \mathbb{N}$, give an example of a sequence of m^2 distinct real numbers with no monotonic subsequence of length $m + 1$.

2.5 Suppose that $P = (X, <)$ is a poset of size less than $\binom{m+2}{2}$. Show that X can be covered by m sets, each of which is either a chain or an antichain in P.

Show that this result is best possible: give an example of a poset of size $\binom{m+2}{2}$ that cannot be covered by m chains and antichains.

2.6 Give a proof of the defect form of Hall's Marriage Theorem using Dilworth's Theorem.

2.7 Observe that all bipartite graphs are perfect. The Perfect Graph Theorem then states that complements of bipartite graphs are perfect. Interpret this result.

2.8 Let P be a poset of width $w \geq 2$. Show that, for every $u < w$, there are u-antichains A and B such that $|A \vee B| > u$. Give examples of posets where

(a) for every pair of antichains A and B, $|A \vee B| \geq \max(|A|, |B|)$,

(b) there is a pair (A, B) of 3-antichains with $|A \vee B| = 2$.

Do the same results hold for $A \wedge B$ as for $A \vee B$?

2.9 Let $P = (X, <)$ be a poset, and take $\mathcal{S} = \{X, U_1, U_2, \ldots, U_k\}$, where $U_1 \subset U_2 \subset \cdots \subset U_k$ is a nested sequence of up-sets. Show that $\mathcal{S}$ has a simultaneous optimal chain partition.

2.10 Give an example of a poset P with two up-sets U_1 and U_2 such that $\{U_1, U_2\}$ does not have a simultaneous optimal chain partition.

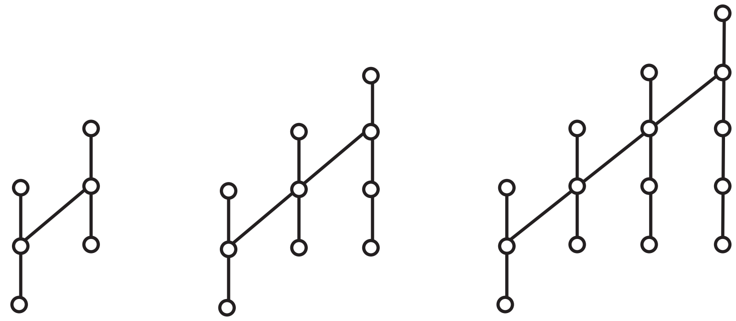

Fig. 2.9. Challenges for Greene-Kleitman

Give an example of a poset $P = (X, <)$ and a subset Y of X such that $\{X, Y\}$ does not have a simultaneous optimal chain partition.

2.11 We show in Figure 11 the first three posets in an infinite sequence $\{P_h : h \geq 4\}$. Show that the conclusion of West's Theorem 2.6.6 holds in the case of the Greene-Kleitman theorem. Then construct complementary posets for Greene's theorem.

2.12 Let U be an up-set and D a down-set in a poset $P = (X, <)$. Show that $\{X, U, D\}$ has a simultaneous optimal chain partition. Show further that, if $\mathcal{S}$ is a nested set of up-sets and $\mathcal{T}$ is a nested set of down-sets, then $\mathcal{S} \cup \mathcal{T}$ has a simultaneous optimal chain partition.

2.13 Let $P = (X, <)$ be a poset, and let B be the complete bipartite graph with two copies of X as vertex classes. Describe how to assign weights to the edges of B so that a maximum-weight matching corresponds naturally (i.e., as in our second proof of Dilworth's Theorem) to a chain partition $\mathcal{C}$ minimizing $e_k(\mathcal{C})$.

2.14 Let $P = (X, <)$ be a poset, with $X = [n]$, and consider the vector space $\mathbb{R}^n$, with standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. For a subset Y of X, let $\langle Y \rangle$ be the subspace of $\mathbb{R}^n$ spanned by the vectors $\mathbf{e}_i$ with $i \in Y$. Define a *down-map* to be a linear map $\phi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\phi(\mathbf{e}_i) \in \langle D(i) \rangle$ for every i (i.e., $\phi(\mathbf{e}_i)$ is a linear combination of the $\mathbf{e}_j$ with $j < i$).

Show that the nullity $n(\phi)$ of a down-map ϕ is at least $w(P)$, and use Dilworth's Theorem to show that there is a down-map ϕ with rank $r(\phi)$ at least $n - w(P)$, (and so $n(\phi) = w(P)$).

Show further, by induction on k, that, for ϕ a down-map, $n(\phi^k) \geq w_k(P)$ for $k \in \mathbb{N}$. Finally, show that, for each k, there is a down-map ϕ with $n(\phi^k) = w_k(P)$.

2.15 A *partition* of the integer n is a non-increasing sequence $n_1 \succeq n_2 \succeq \cdots \succeq n_t \succ 0$ of positive integers summing to n. Two partitions $(n_1, \ldots, n_t)$ and $(m_1, \ldots, m_s)$ of the same integer n are *conjugate* if, for each $i = 1, \ldots, s$, the largest value of j such that $n_j \geq i$ is exactly m_i. Show that this implies that, for each j, the largest value of i such that $m_i \geq j$ is exactly n_j.

Show that, for any n-element poset P of height s and width t, the sequences $(w_1 - w_0, w_2 - w_1, \ldots, w_s - w_{s-1})$ and $(h_1 - h_0, h_2 - h_1, \ldots, h_t - h_{t-1})$ are conjugate partitions of n. (Here $w_i = w_i(P)$ and $h_i = h_i(P)$, for all i.)

2.16 Let k and m be positive integers, and let P be an n-element poset of height at least k and width at least m. Set a, b, c and d to be the integers such that $h_m(P) = am + b$, $0 \leq b < m$, $w_k(P) = ck + d$, and $0 \leq d < k$. Show that

$$n \leq \max(ac + b + d, h_m(P) + w_k(P) - k - m + 1).$$

Show further that this inequality is best possible for all values of k, m, h_m and w_k.

Hint: fix k, choose r such that $(k, r) \in \Gamma(P)$, and use an argument similar to that in the proof of Theorem 2.9.4.

2.17 Let s, t and p be integers with $p \leq \min(\lfloor (s+1)/2 \rfloor, \lfloor (t+1)/2 \rfloor)$, and let P be any poset with $n \geq st + 2p - 1$ elements. Use the previous exercise to show that there are either $\lfloor (s+1)/2 \rfloor$ chains covering at least $\lfloor (s+1)/2 \rfloor t + p$ elements of P, or $\lfloor (t+1)/2 \rfloor$ antichains covering at least $\lfloor (t+1)/2 \rfloor s + p$ elements of P.

2.11 Notes and References

Theorem **??** was apparently first noted in print by Mirsky [**?**] in 1971, although by then there were several proofs of Dilworth's Theorem in the literature, and the result was undoubtedly known to many well before then.

Researchers in discrete mathematics are likely to come across references to work of Erdős and Szekeres from 1935 in several different contexts: in that year they published fundamental results in Ramsey theory (upper bounds on Ramsey numbers) and in combinatorial geometry (existence of and estimates for a function $f(n)$ such that any $f(n)$ points in general position in the plane contain n that form a convex n-gon), besides the result appearing here as Theorem **??**. What is astonishing is that all of these results appear in one 8-page paper [**?**]: a classic!

Proofs of Dilworth's Theorem are many and various. Besides the two proofs we give here, due to Perles [**?**] and to Fulkerson [**?**], and Dilworth's original [**?**], we mention a variant of Perles' proof due to Tverberg [**?**], and a proof published by Galvin [**?**] in 1994 (which he admits is likely to be a rediscovery) that is essentially an inductive version of our proof of Lemma **??**.

The *Handbook of Combinatorics* contains many excellent survey articles: ones particularly relevant to material here are the chapters touching on Matchings (by Pulleyblank [**?**]), on Perfect Graphs (by Toft [**?**]) and on Network Flows (by Frank [**?**]).

The Greene-Kleitman Theorem appears in a paper [**?**] in JCT(A), in 1976. The paper following that in the journal is by Greene [**?**], proving Corollary **??** and therefore, explicitly or implicitly, all of the other results in Section **??**. (The next paper in the journal is also by Greene and Kleitman, and the one after has Kleitman as a co-author). The version of Theorem **??** based on conjugate partitions of n (see Exercise 15) appears in another paper of Greene [**?**] that, despite appearing in 1974, refers to the 1976 paper of Greene and Kleitman.

The proof of the Greene-Kleitman Theorem we give here is due to Saks [**?**]. In that paper, he only proves that, for each k, there is some k-full family of chains. The extension of his proof to give the full result appears in his PhD thesis [**?**], and also in a paper of Perfect [**?**]. Exercise 14 is based on another result of Saks [**?**]; what we term a *down-map* is more properly known as a member of the *strict incidence algebra* of P.

The Greene-Kleitman Theorem and the related results have the flavor of network flows, or more generally of the duality theory of linear programming, and indeed there are ways to prove the results via these techniques. Frank [**?**] (or see his Chapter [**?**] in the *Handbook of Combinatorics*) gives a direct proof of Corollary 2.7.3, and hence of the Greene-Kleitman Theorem and all the other consequences, by analysing the behavior of the Ford-Fulkerson min-cost-flow algorithm on a suitably designed network, as the required amount of flow is gradually increased. This is undoubtedly a more streamlined proof than the one given here, but we chose to give a proof more firmly grounded in the combinatorial theory of posets. A similar proof was given independently by Fomin [**?**].

Various generalizations and extensions of the Greene-Kleitman theorem are surveyed in a long article by West [**?**]. Although dating from 1985, this contains references to almost all of the important material on this topic.

2.12 Things Left to Do

1. I haven't checked GRB's argument for Greene's theorem carefully, i.e., the part where he uses network flows to show that $h_{w+1} - h_w \leq h_w - h_{w-1}$.
2. The references are incomplete and perhaps the attributions need to be looked at again.
3. The historical comments needs to be updated in view of the edits and additional material.
4. The exercises should be revisited.
5. A number of references from GRB's portion are now undefined.

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