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CHAPTER 8

Partially Ordered Sets

William T. TROTTER

Department of Mathematics, Arizona State University, Tempe AZ 85287, USA

Contents

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Introduction

Interest in finite partially ordered sets has been heightened in recent years by a steady stream of theorems combining clever ad hoc arguments with powerful techniques from other areas of mathematics. In this chapter, we present a sampling of results exhibiting these characteristics. In those instances where we do not present a complete proof, we outline enough of the general contours of the argument to allow the reader to supply the missing details with little difficulty. We also outline anticipated research directions in the combinatorics of partially ordered sets, and we discuss briefly some of the most interesting open problems in this field.

Since this Handbook contains chapters on Extremal Set Theory and Enumeration, we have limited our discussion to results on general partially ordered sets. Still some difficult choices had to be made concerning results to be included – especially in view of our emphasis on proof techniques. West's survey articles (West 1982, 1985) offer more of a catalogue of theorems in the area and have extensive bibliographies. Also, we recommend the recent books by Anderson (1987), Fishburn (1986), Stanley (1986), and Trotter (1992) as well as the conference volumes (Rival 1982, 1985) for additional material on partially ordered sets and related topics.

1. Notation and terminology

Formally, a *partially ordered set* is a pair (X, P) where X is a set, and P is a reflexive, antisymmetric, and transitive binary relation on X. The set X is called the *ground set* and P is called a *partial order*. Throughout this chapter, we use the short form *poset* for a partially ordered set. Many researchers choose to drop the adjective "partially" and use *ordered set* to mean a poset. A poset (X, P) is *finite* if the ground set X is finite. In this chapter, we will be concerned primarily with finite posets.

In some settings, we find it convenient to use a single symbol such as $\mathbf{P}$ to denote a poset (X, P). This notation is particularly handy when both the ground set X and the partial order P remain fixed. In other settings, especially when we have several partial orders on the same ground set, we will use the ordered pair notation for posets.

The notations $(x, y) \in P$, xPy, $x \leqslant y$ in P, and $y \geqslant x$ in P are used interchangeably. The notation $x < y$ in P means $x \leqslant y$ in P and $x \neq y$. Distinct points x, y are *comparable* when either $x < y$ or $y < x$ in P. Otherwise, we say x and y are *incomparable* and write $x \| y$ in P. When using a single symbol like $\mathbf{P}$ for a poset, we will write $x < y$ in $\mathbf{P}$, $x \| y$ in $\mathbf{P}$, etc.

A poset $\mathbf{P} = (X, P)$ is a *chain* (also a *totally ordered set* or a *linearly ordered set*) if each pair of distinct points is comparable. We will use the symbols $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ to denote the *reals*, *rationals*, *integers* and *positive integers*, respectively. Each of these posets is a chain.

Dually, $\mathbf{P} = (X, P)$ is an *antichain* if each pair of distinct points is incomparable.

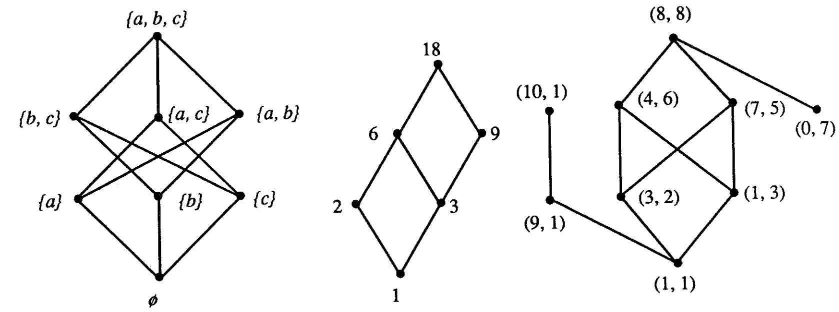

Figure 1.1.

If $Y \subset X$ and Q is the restriction of P to Y, then the poset $\mathbf{Q} = (Y, Q)$ is called a *subposet* of (X, P). A subset $Y \subset X$ is also called a *chain* (*antichain*) if the subposet (Y, Q) is a chain (antichain). The *height* of a poset is the maximum cardinality of a chain, and the *width* is the maximum cardinality of an antichain.

When $\mathbf{P} = (X, P)$ and $\mathbf{Q} = (Y, Q)$ are posets, a map $f: X \to Y$ is called an *embedding* (of $\mathbf{P}$ into $\mathbf{Q}$) if $x_1 \leqslant x_2$ in $P \iff f(x_1) \leqslant f(x_2)$ in Q. An embedding $f: X \to Y$ is an *isomorphism* when $f(X) = Y$. In this chapter, we prefer not to distinguish between isomorphic posets and to write $\mathbf{P} = \mathbf{Q}$ to indicate that the two posets are isomorphic. Similarly we say that $\mathbf{P}$ is *contained* in $\mathbf{Q}$ (also $\mathbf{P}$ is a *subposet* of $\mathbf{Q}$) when there exists an embedding of $\mathbf{P}$ in $\mathbf{Q}$.

We say *y covers x* in P and write $x <: y$ in P when there is no z for which both $x < z$ and $z < y$ in P. The *cover graph* associated with the poset $\mathbf{P} = (X, P)$ is the graph $\mathbf{G} = (X, E)$ whose edge set E consists of the pairs xy for which $x <: y$ in P. A drawing of the cover graph $\mathbf{G} = (X, E)$ in the Euclidean plane is called a *Hasse diagram* (or *order diagram*) of the poset $\mathbf{P} = (X, P)$ if x is lower in the plane than y whenever $x <: y$ in P.

Here are some frequently encountered examples of posets. Any family of sets is partially ordered by set inclusion; a set of positive integers is partially ordered by division; and a subset of $\mathbb{R}^n$ is partially ordered by $(a_1, a_2, \ldots, a_n) \leqslant (b_1, b_2, \ldots, b_n) \iff a_i \leqslant b_i$ in $\mathbb{R}$ for $i = 1, 2, \ldots, n$. In fig. 1.1, we show particular instances of these examples. Each has height 4, and their respective widths are 3, 2, and 4.

If P and Q are partial orders on the same ground set X, Q is called an *extension* of P when $P \subseteq Q$. The partial order Q is called a *linear extension* of P if Q is an extension of P and (X, Q) is a chain.

When $\mathbf{P} = (X, P)$ is a poset, an element $x \in X$ is called a *maximal* (*minimal*) element if there is no $y \in X$ for which $x < y$ in P ($y < x$ in P). The set of maximal (minimal) elements is denoted $\mathrm{MAX}(X, P)$ ($\mathrm{MIN}(X, P)$). The subsets $\mathrm{MAX}(X, P)$

and $\mathrm{MIN}(X,P)$ always determine antichains, although neither may be as large as the width of (X,P).

When $Y \subset X$, the set $\{z \in X\colon y \leqslant z \text{ in } P \text{ for every } y \in Y\}$ is called the set of *upper bounds* for Y. Note that this set may be empty. When the set of upper bounds of Y is nonempty and has a least element, this unique point is called the *least upper bound* of Y and is denoted $\mathrm{l.u.b.}(Y)$. Dually, the *greatest lower bound* (if it exists) of Y is denoted $\mathrm{g.l.b.}(Y)$.

A poset $\mathbf{P} = (X,P)$ is called a *lattice* when each nonempty subset $Y \subset X$ has both a least upper bound and a greatest lower bound. When $\mathbf{P} = (X,P)$ is a lattice and $x,y \in X$, we write $x \vee y$ for $\mathrm{l.u.b.}\{x,y\}$ and $x \wedge y$ for $\mathrm{g.l.b.}\{x,y\}$. The binary operations $\vee$ (join) and $\wedge$ (meet) are commutative and associative. The lattice is *distributive* if $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ for all $x,y,z \in X$.

When $\mathbf{P}$ and $\mathbf{Q}$ are posets, the *disjoint sum* of $\mathbf{P}$ and $\mathbf{Q}$, denoted $\mathbf{P}+\mathbf{Q}$, is obtained by taking the union of disjoint copies of the two posets with no comparabilities between the points in one and points in the other. A poset is *disconnected* if it is the disjoint sum of two proper subposets; otherwise it is *connected.* The maximal connected subposets of a disconnected poset are *components.*

The *cartesian product* of $\mathbf{P} = (X,P)$ and $\mathbf{Q} = (Y,Q)$, denoted $\mathbf{P} \times \mathbf{Q}$, consists of the ordered pairs (x,y) where $x \in X$ and $y \in Y$ with partial ordering $(x_1,y_1) \leqslant (x_2,y_2) \iff x_1 \leqslant x_2$ in $\mathbf{P}$ and $y_1 \leqslant y_2$ in $\mathbf{Q}$. The cartesian product of n copies of $\mathbf{P}$ is denoted $\mathbf{P}^n$.

Given posets $\mathbf{P} = (X,P)$ and $\mathbf{Q} = (Y,Q)$, a function $f\colon X \to Y$ is an *order preserving* (or *monotone*) map from $\mathbf{P}$ to $\mathbf{Q}$ if $x_1 \leqslant x_2$ in $P \Rightarrow f(x_1) \leqslant f(x_2)$ in Q. The set of all order preserving maps from $\mathbf{P}$ to $\mathbf{Q}$ is partially ordered by $f_1 \leqslant f_2 \iff f_1(x) \leqslant f_2(x)$ in Q for every $x \in X$. This poset is denoted $\mathbf{Q}^{\mathbf{P}}$.

Throughout the chapter, we use $\mathbf{k}$ to denote a k-element chain $0 < 1 < 2 < \cdots < k-1$. The poset $\mathbf{2}^n$ is isomorphic to the set of subsets of an n-element set partially ordered by inclusion. A poset $\mathbf{P}$ is a distributive lattice if and only if there is a poset $\mathbf{Q}$ so that $\mathbf{P}$ is isomorphic to $\mathbf{2}^{\mathbf{Q}}$ [see chapter 3 in Birkhoff (1973)].

When $\mathbf{P} = (X,P)$ is a poset and $\mathscr{F} = \{\mathbf{P}_x = (Y_x,Q_x)\colon x \in X\}$ is a family of posets indexed by the ground set of $\mathbf{P}$, the *lexicographic sum* of $\mathscr{F}$ over $\mathbf{P}$ is the poset whose ground set is $\{(x,y)\colon x \in X, y \in Y_x\}$. The partial ordering is defined by $(x_1,y_1) \leqslant (x_2,y_2) \iff (x_1 < x_2$ in $P)$ or $(x_1 = x_2$ and $y_1 \leqslant y_2$ in $Q_{x_1})$. A lexicographic sum is nontrivial if $|X| \geqslant 2$ and if at least one Y_x satisfies $|Y_x| \geqslant 2$. A poset $\mathbf{P}$ is *decomposable* if it is isomorphic to a nontrivial lexicographic sum; otherwise $\mathbf{P}$ is *indecomposable.* Note that the disjoint sum of two posets is a lexicographic sum over a 2-element antichain.

2. Dilworth's theorem and the Greene–Kleitman theorem

Dilworth's decomposition theorem (Dilworth 1950) has played an important role in motivating research in posets, as evidenced by results discussed in this section as well as in sections 3, 6, 7 and 8. Also, Dilworth's theorem surfaces in a variety of extremal problems (see, for example, Duffus et al. 1991). There are several elementary proofs; the one we present is patterned after Perles (1963).

Theorem 2.1. *If* $\mathbf{P} = (X, P)$ *is a poset of width n, then there exists a partition* $X = C_1 \cup C_2 \cup \cdots \cup C_n$ *where each* C_i *is a chain.*

Proof. We proceed by induction on $|X|$ and note that the result is trivial when $|X| = 1$. Assume validity when $|X| < k$ and consider a poset $\mathbf{P}$ with $|X| = k$. We may assume that the width n of $\mathbf{P}$ is larger than 1.

Choose $x \in \mathrm{MAX}(\mathbf{P})$ and $y \in \mathrm{MIN}(\mathbf{P})$ with $y \leqslant x$. Let $\mathbf{Q}$ be the poset obtained by removing x and y from $\mathbf{P}$. If the width of $\mathbf{Q}$ is less than n, then we can partition $\mathbf{Q}$ into fewer than n chains which together with the chain $\{x, y\}$ form a partition of X into (at most) n chains. So we may assume that $\mathbf{Q}$ has width n. Thus $y < x$ in $\mathbf{P}$. Choose an n-element antichain $A = \{a_1, a_2, \ldots, a_n\}$ in $\mathbf{Q}$.

Then let $U = \{u \in X\colon u \geqslant a_i \text{ for some } a_i \in A\}$ and $D = \{d \in X\colon d \leqslant a_j \text{ for some } a_j \in A\}$. Evidently $x \in U - D$ and $y \in D - U$. Thus there are chain partitions $U = C_1' \cup C_2' \cup \cdots \cup C_n'$ and $D = C_1'' \cup C_2'' \cup \cdots \cup C_n''$. We may label these chains so that $a_i \in C_i' \cap C_i''$ for $i = 1, 2, \ldots, n$. Then $C_i = C_i' \cup C_i''$ is a chain for each i and the desired partition is $X = C_1 \cup C_2 \cup \cdots \cup C_n$. □

In introductory combinatorics texts, Dilworth's theorem is grouped with other max–min theorems having a common theme: P. Hall's marriage theorem, the König–Egervary theorem, Menger's theorem, and the max flow–min cut theorem for network flows. This last result most clearly captures the linear programming core common to all. (See chapters 2 and 3 by Frank and Pulleyblank for additional material.)

Dilworth's theorem has a trivial dual version for antichains.

Theorem 2.2. *If* $\mathbf{P} = (X, P)$ *is a poset of height n, then there exists a partition* $X = A_1 \cup A_2 \cup \cdots \cup A_n$ *where each* A_i *is an antichain.*

Proof. Set $A_1 = \mathrm{MAX}(\mathbf{P})$. Thereafter set $A_{i+1} = \mathrm{MAX}(\mathbf{P}_i)$ where $\mathbf{P}_i$ is the subposet obtained by removing the antichains $A_1, A_2, \ldots, A_i$ from $\mathbf{P}$. □

The first major result in this chapter is an important generalization of Dilworth's chain partitioning theorem due to Greene and Kleitman (1976). The proof we give here is patterned after algorithmic proofs given by Saks (1979) and Perfect (1984). An alternative proof using network flows is given in this volume in chapter 2.

We need some preliminary notation and terminology. Let $\mathbf{P} = (X, P)$ be a poset and k a positive integer. A subset $S \subset X$ is called a *Sperner k-family* if S does not contain a chain of $(k+1)$-elements. The maximum cardinality of a Sperner k-family is denoted $d_k(\mathbf{P})$. When $\mathscr{C} = \{C_1, C_2, \ldots, C_t\}$ is a family of chains forming a partition of X, we define $e_k(\mathscr{C}) = \sum_{i=1}^{t} \min\{k, |C_i|\}$. If S is any Sperner k-family and $\mathscr{C} = \{C_1, C_2, \ldots, C_t\}$ is any chain partition of X, we note that $|S \cap C_i| \leqslant \min\{k, |C_i|\}$. Thus $|S| \leqslant e_k(\mathscr{C})$, so that $d_k(\mathbf{P}) \leqslant e_k(\mathscr{C})$. The chain partition $\mathscr{C}$ is said to be *k-saturated* if $d_k(\mathbf{P}) = e_k(\mathscr{C})$.

We also need a prelimi
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Lemma 2.3. *The set* $\mathscr{M}$
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With this background,

Theorem 2.4. *Let* $\mathbf{P}$ *be*
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Proof. We first show th
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We also need a preliminary lemma whose elementary proof is omitted. Let $\mathcal{M}(\mathbf{P})$ denote the set of all maximum antichains of $\mathbf{P}$. Define a partial order on $\mathcal{M}(\mathbf{P})$ by $A \leqslant B \iff$ for every $a \in A$, there exists $b \in B$ with $a \leqslant b$.

Lemma 2.3. *The set $\mathcal{M}(\mathbf{P})$ of maximum antichains of a poset $\mathbf{P} = (X, P)$ has a unique greatest element.*

With this background, here is the Greene–Kleitman theorem.

Theorem 2.4. *Let $\mathbf{P}$ be a poset and k a positive integer. Then there exists a chain partition $\mathscr{C}$ of $\mathbf{P}$ which is simultaneously k-saturated and $(k+1)$-saturated, i.e., $d_k(\mathbf{P}) = e_k(\mathscr{C})$ and $d_{k+1}(\mathbf{P}) = e_{k+1}(\mathscr{C})$.*

Proof. We first show that $d_1(\mathbf{P} \times \mathbf{k}) = d_k(\mathbf{P})$ for every $k \geqslant 1$. Let A be a maximum antichain in $\mathbf{P} \times \mathbf{k}$, and let $A_i = \{x \in X\colon (x, i) \in A\}$. Then each A_i is an antichain in $\mathbf{P}$, so the set $S = A_1 \cup A_2 \cup \cdots \cup A_k$ is a Sperner k-family. Furthermore, $|S| = |A|$ since $A_i \cap A_j = \emptyset$ when $i \neq j$. Thus $d_1(\mathbf{P} \times \mathbf{k}) \leqslant d_k(\mathbf{P})$.

Conversely, let S be a maximum Sperner k-family in $\mathbf{P}$. Partition S into k antichains by setting $A_1 = \mathrm{MAX}(S)$ and $A_{i+1} = \mathrm{MAX}(S - (A_1 \cup A_2 \cup \cdots \cup A_i))$. Then $A = \{(a, i)\colon a \in A_i\}$ is an antichain in $\mathbf{P} \times \mathbf{k}$ with $|A| = |S|$. This shows $d_1(\mathbf{P} \times \mathbf{k}) \geqslant d_k(\mathbf{P})$. Thus $d_1(\mathbf{P} \times \mathbf{k}) = d_k(\mathbf{P})$.

For the remainder of the proof, we fix a positive integer k. Then we make several definitions concerning chain partitions of $\mathbf{P} \times (\mathbf{k+1})$. When $\mathscr{C} = \{C_1, C_2, \ldots, C_t\}$ is a chain partition of $\mathbf{P} \times (\mathbf{k+1})$, we let $M(\mathscr{C}) = \{\mathrm{MAX}(C_i)\colon 1 \leqslant i \leqslant t\}$. We say *$y$ covers x* in $\mathscr{C}$ if there is some $C_i \in \mathscr{C}$ so that y covers x in the chain C_i. When $S \subset \mathbf{P} \times (\mathbf{k+1})$, the set $\{x \in X\colon (x, i) \in S$ for some $i\}$ is called the *projection* of S on $\mathbf{P}$. For each i, the subset $S \cap (\mathbf{P} \times \{i\})$ is called *level i* of S. The projection on $\mathbf{P}$ of level i of $M(\mathscr{C})$ is denoted by $M_i(\mathscr{C})$.

A chain partition $\mathscr{C}$ of $\mathbf{P} \times (\mathbf{k+1})$ is *special* if the following two conditions hold:

(i) $M_0(\mathscr{C}) \supset M_1(\mathscr{C}) \supset M_2(\mathscr{C}) \supset \cdots \supset M_{k-1}(\mathscr{C})$;

(ii) If $x \in M_k(\mathscr{C}) - M_{k-1}(\mathscr{C})$, then (x, k) covers $(x, k-1)$ in $\mathscr{C}$.

A special chain partition of $\mathbf{P} \times \mathbf{k+1}$ is *very special* if it also satisfies the following two conditions:

(iii) Exactly $d_{k+1}(\mathbf{P}) - d_k(\mathbf{P})$ of the chains in $\mathscr{C}$ are subsets of level 0; and

(iv) $|\mathscr{C}| = d_1(\mathbf{P} \times (\mathbf{k+1}))$.

When $\mathscr{C}$ is special, it follows from the second condition in this definition that $M_k(\mathscr{C}) = N_1(\mathscr{C}) \cup N_2(\mathscr{C})$ where $N_1(\mathscr{C}) = M_k(\mathscr{C}) \cap M_{k-1}(\mathscr{C})$. If $x \in N_2(\mathscr{C})$, then (x, k) covers $(x, k-1)$ in $\mathscr{C}$.

We now show that the theorem follows whenever $\mathbf{P} \times (\mathbf{k+1})$ has a very special chain partition. To see this, let $\mathscr{C}_{k+1} = \{C_1, C_2, \ldots, C_t\}$ be a very special chain partition of $\mathbf{P} \times (\mathbf{k+1})$ where $t = d_1(\mathbf{P} \times (\mathbf{k+1}))$. Set $s = d_{k+1}(\mathbf{P}) - d_k(\mathbf{P})$. We assume that $C_1, C_2, \ldots, C_s$ are subsets of level 0. For each $j = 1, 2, \ldots, t$, let $D_j = \{(x, i)\colon (x, i+1) \in C_j\}$. Of course, $D_1, D_2, \ldots, D_s$ are all empty. Let $\mathscr{C}_k$ be the collection of all nonempty D_j's. Then $\mathscr{C}_k$ is a chain partition of $\mathbf{P} \times \mathbf{k}$ and $|\mathscr{C}_k| \leqslant t - s = d_k(\mathbf{P}) = d_1(\mathbf{P} \times \mathbf{k})$. Thus $|\mathscr{C}_k| = d_k(\mathbf{P})$. Furthermore, it is easy to see that $\mathscr{C}_k$ is a special chain partition of $\mathbf{P} \times \mathbf{k}$.

Now level k in $\mathbf{P} \times (\mathbf{k}+\mathbf{1})$ forms a copy of $\mathbf{P}$ and the $|M_k(\mathscr{C}_{k+1})|$ chains in $\mathscr{C}_{k+1}$ which intersect level k determine a chain partition of $\mathbf{P}$ which we denote by $\mathscr{C}$. We show that $\mathscr{C}$ is both k-saturated and $(k+1)$-saturated.

Now let $j \in \{k, k+1\}$. Then

$$\begin{aligned} d_j(\mathbf{P}) = e_1(\mathscr{C}_j) = |M(\mathscr{C}_j)| &= \sum_{i=0}^{j-1} |M_i(\mathscr{C}_j)| \\ &\geqslant (j-1)|M_{j-2}(\mathscr{C}_j)| + |M_{j-1}(\mathscr{C}_j)| \\ &= (j-1)|M_{j-2}(\mathscr{C}_j)| + |M_{j-1}(\mathscr{C}_j) \cap M_{j-2}(\mathscr{C}_j)| + |N_2(\mathscr{C}_j)| \\ &\geqslant j|M_{j-1}(\mathscr{C}_j) \cap M_{j-2}(\mathscr{C}_j)| + |N_2(\mathscr{C}_j)| \\ &= j|N_1(\mathscr{C}_j)| + |N_2(\mathscr{C}_j)| \\ &\geqslant \sum_{E \in \mathscr{E}} \min\{j, |E|\} \\ &= e_j(\mathscr{E}) \geqslant d_j(\mathbf{P}). \end{aligned}$$

Thus $\mathscr{E}$ is both k-saturated and $(k+1)$-saturated as claimed. To complete the proof, we need only show the existence of a very special chain partition of $\mathbf{P} \times (\mathbf{k}+\mathbf{1})$. Set $t = d_{k+1}(\mathbf{P}) = d_1(\mathbf{P} \times (\mathbf{k}+\mathbf{1}))$. Of all partitions of $\mathbf{P}(k+1)$ into t chains, choose one having as many chains as possible as subsets of level 0. Call this partition $\mathscr{C} = \{C_1, C_2, \ldots, C_t\}$ and label the chains in $\mathscr{C}$ so that $C_1, C_2, \ldots, C_s$ are subsets of level 0 but $C_{s+1}, C_{s+2}, \ldots, C_t$ are not. Since the last $t-s$ chains in $\mathscr{C}$ cover a copy of $\mathbf{P} \times \mathbf{k}$, we know $t - s \geqslant d_1(\mathbf{P} \times \mathbf{k}) = d_k(\mathbf{P})$, so $s \leqslant d_{k+1}(\mathbf{P}) - d_k(\mathbf{P})$. We show that $s = d_{k+1}(\mathbf{P}) - d_k(\mathbf{P})$. Suppose to the contrary that $s < d_{k+1}(\mathbf{P}) - d_k(\mathbf{P})$.

Let $\mathbf{Q} = \mathbf{P} \times (\mathbf{k}+\mathbf{1}) - (C_1 \cup C_2 \cup \cdots \cup C_s)$. Clearly, the width of $\mathbf{Q}$ is $t-s$. Let A be the unique greatest element in the poset $\mathscr{M}(\mathbf{Q})$ of maximum antichains in $\mathbf{Q}$. A contains at least $d_{k+1}(\mathbf{P}) - d_k(\mathbf{P}) - s$ elements from level 0 since the width of the top k levels is only $d_k(\mathbf{P})$. Choose an element $a_0 \in A$ which comes from level 0. Without loss of generality $a_0 \in C_{s+1}$. Let $\mathbf{Q}' = \mathbf{Q} - \{c \in C_{s+1} \colon c \leqslant a_0\}$.

We claim that the width of $\mathbf{Q}'$ is less than $t-s$, for if $\mathbf{Q}'$ contains a $(t-s)$-element antichain B, then B contains an element b with $a_0 < b$ in $\mathbf{Q} \times (\mathbf{k}+\mathbf{1})$. This contradicts our choice of A. It follows that we can partition Y' into $t-s-1$ chains which together with $C_1, C_2, \ldots, C_s$ and $\{c \in C_{s+1} \colon c \leqslant a_0\}$ form a partition of $\mathbf{P} \times (\mathbf{k}+\mathbf{1})$ into t chains. In this partition, there are $s+1$ chains which are subsets of level 0. The contradiction shows $s = d_{k+1}(\mathbf{P}) - d_k(\mathbf{P})$.

We now proceed to transform $\mathscr{C}$ into a very special partition by a series of operations called *insertions* and *switches*. At this moment $\mathscr{C}$ satisfies properties (iii) and (iv), and both operations preserve these properties.

We first perform a series of insertions. Choose points (x,i), (y,j) so that (x,i) covers (y,j) in $\mathscr{C}$ and $i > j$. If $i \neq j+1$ or $x \neq y$, remove $(y, j+1)$ from the chain to which it currently belongs and insert it in the chain containing (x,i) and (y,j). Repeat until no further insertions are possible.

Next, we perform a seri
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Corollary 2.5. *Let* $\mathbf{P}$ *be*
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3. Kierstead's chain parti

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Next, we perform a series of switches. For an integer $j \geqslant 1$, locate a point $(x, j) \in M(\mathscr{C})$ so that either: (1) $j < k$ and $(x, j-1) \notin M(\mathscr{C})$; or (2) $j = k$ and (x, j) does not cover $(x, j-1)$ in $\mathscr{C}$, and $(x, j-1) \notin M(\mathscr{C})$.

Let (y, i) be the point covering $(x, j-1)$ and let C be the chain containing $(y, i+1)$. Let C'' consist of those points in $\mathscr{C}$ which are less than $(y, i+1)$ and let $C'' = C - C'$. Then let D be the chain containing (x, j) and set $D' = D \cup C''$. Replace C and D in $\mathscr{C}$ by C' and D'. Repeat until no further switches are possible.

It is obvious that the series of insertions must stop, but it takes a moment's reflection to see that this is also true for the series of switches. For $j = 1, 2, \ldots, k$, let v_j count the number of points $x \in X$ for which (x, j) covers (y, i) in $\mathscr{C}$ and $(x, j-1)$ covers $(y, i-1)$ in $\mathscr{C}$. Each time we perform a switch, the vector $(v_1, v_2, \ldots, v_k)$ increases lexicographically. Since $v_j \leqslant |X|$ for each j, the procedure stops. □

This theorem has many significant applications. The following corollary follows easily, but we know of no simple proof avoiding the use of Theorem 2.4.

Corollary 2.5. *Let* $\mathbf{P}$ *be a finite poset. Then for each* $k \geqslant 1$, $d_k(\mathbf{P}) - d_{k-1}(\mathbf{P}) \geqslant d_{k+1}(\mathbf{P}) - d_k(\mathbf{P})$.

3. Kierstead's chain partitioning theorem

In this section we outline the proof of a theorem of Kierstead (1981) which asserts that for each $n \geqslant 1$, there is a $t = t(n)$ for which there exists an on-line algorithm which will partition any poset $\mathbf{P}$ of width at most n into t chains. By an on-line partition, we mean that the poset and the partition are constructed one point at a time. An adversary (infinitely clever) constructs the poset and we must devise the partition. At each round, the adversary presents the new point and describes its comparabilities and incomparabilities to all preceding points. We must then add the new point to one of the sets making up the partition. Both players' moves are permanent.

As a warm-up, we first present the on-line version of the dual to Dilworth's theorem. The result is an unpublished theorem of Schmerl, although a short proof is given in Kierstead (1986).

Theorem 3.1. *For each* $n \geqslant 1$, *there exists an algorithm which will construct an on-line partition of a poset of height at most* n *into* $n(n+1)/2$ *antichains.*

Proof. When the new point x is added to the poset, let $r = r(x)$ be the maximum number of points in a chain having x as least element, and let $s = s(x)$ be the maximum number of points in a chain having x as greatest element. Assign x to the set $A(r, s)$. Clearly, each $A(r, s)$ is an antichain. Since $r + s - 1 \leqslant n$, there are $n(n+1)/2$ such sets. □

Szemerédi produced a simple argument to show that Theorem 3.1 is best possible, and we invite the reader to reconstruct his proof. Full details are given in

Kierstead (1986). As a first step, show that there exists a strategy for constructing a poset **P** of height at most n which will force any opponent producing an on-line partition into antichains to use at least $n(n+1)/2$ antichains in covering **P** and at least n antichains in covering MAX(**P**).

Here is Kierstead's on-line chain partitioning theorem (Kierstead 1981).

Theorem 3.2. *For each $n \geqslant 1$, there exists an algorithm which will construct an on-line partition of a poset of width at most n into $(5^n - 1)/4$ chains.*

Proof. The argument proceeds by induction on n with the case $n = 1$ being trivial. The heart of the argument is the case $n = 2$ where we have to partition a width-2 poset into 6 chains.

We first construct a greedy chain C_1. As a new point enters the poset, we insert it in C_1 whenever it is comparable to all other points previously placed in C_1. Thus for every $x \in X - C_1$, there is a nonempty set $I(x)$ of points from C_1 which are incomparable to x. Although $I(x)$ may grow with time, it is always a set of consecutive points from C_1. When $x, y \in X - C_1$, we write $I(x) < I(y)$ when $u < v$ for every $u \in I(x)$ and every $v \in I(y)$. Note that if x and y are incomparable points in $X - C_1$, then the following condition holds:

(K). When the latter of x and y enters, either

$$I(x) < I(y) \text{ or } I(y) < I(x).$$

In fact, when $n = 2$, the qualifying phrase "when the latter of x and y enters" can be dropped since $I(x) \cap I(y) = \emptyset$ whenever $x \parallel y$. Regardless, we choose the weaker statement since it is crucial to the inductive step.

We define a partial order, called the $*$-order, on $X - C_1$ as follows. When the new point x enters, we set $x * y$ if

(1) $x < y$ in $\mathbf{P} - C_1$, or

(2) $x \parallel y$ and $I(x) < I(y)$.

Similarly, we set $y * x$ if

(3) $y < x$ in $\mathbf{P} - C_1$, or

(4) $x \parallel y$ and $I(y) < I(x)$.

With this definition, it is straightforward to verify that $(X - C_1, *)$ is a chain, i.e., $*$ is a linear extension of the original partial order on $X - C_1$. Next, we define an equivalence relation on $\mathbf{P} - C_1$. Just as is the case with the $*$-order, the definition of this equivalence relation is on-line. The relation will satisfy:

(a) each equivalence class is a set of consecutive elements of $X - C_1$ in the $*$-order, and

(b) if x and y are consecutive elements belonging to the same equivalence class, then $I(x) \cap I(y) \neq \emptyset$.

When a new point x enters $X - C_1$, we put x in the same equivalence class as y if $x <: y$ in $*$ and $I(x) \cap I(y) \neq \emptyset$. If no such y exists, we put x in the same class as z if $z <: x$ in $*$ and $I(x) \cap I(z) \neq \emptyset$. If neither of these results in the assignment of x to an existing class, start a new equivalence class whose only element (at this moment) is x.

Note that if x enters between two consecutive points y and z belonging to the same class, then $I(x) \cap I(y) \neq \emptyset \neq I(x) \cap I(z)$. This insures that property (b) will be preserved when x is added to this class.

To complete the proof of the width-2 case, we need to verify the following claim whose proof we leave as an exercise.

Claim. *If S_1 and S_2 are equivalence classes of $X - C_1$, and there are at least two other equivalence classes between them in the $*$-order, then $S_1 \cup S_2$ is a chain in* **P**.

Once the chain is verified, we may use it to devise a simple strategy for constructing an on-line partition $X - C_1 = C_2 \cup C_3 \cup \cdots \cup C_6$. Each of these five chains is the union of equivalence classes, and any two classes which are subsets of the same chain have at least two other classes between them in the $*$-order. When we start a new class for a point x, we assign it to a chain which does not contain any of the four classes – two above x and two below x in the $*$-order.

To obtain the general result when the width of **P** is $n \geqslant 3$, start by constructing a greedy chain C_1 just as before. Then define a partial order $*$ on $X - C_1$ so that $*$ is an extension of the original order on $X - C_1$ and the width of $(X - C_1, *)$ is $n - 1$. When the new point x enters $X - C_1$, we set $x * y$ if either (1) or (2) holds. However, we also set $x * y$ if:

(1′) there exists $u \in X - C_1$ so that $x < u$ in **P** and $u * y$, or

(2′) there exists $v \in X - C_1$ so that $x * v$ and $v < y$ in **P**.

It is easy to see that this more general definition of $*$ is necessary in order to insure that $*$ is transitive. The definition of when $y * x$ must be expanded analogously. With these observations, it is clear that the width of $(X - C_1, *)$ is at most $n - 1$, for if $A = \{a_1, a_2, \ldots, a_m\}$ is an antichain in $(X - C_1, *)$, then $\bigcap_{i=1}^{m} I(a_i) \neq \emptyset$. If $x \in C_1$ and $x \parallel a_i$ for $i = 1, 2, \ldots, m$, then $\{x\} \cup A$ is an antichain in **P**. It follows that there is a strategy for partitioning $X - C_1$ into $(5^{n-1} - 1)/4$ subsets each of which is a chain in the $*$-order. Observe that a $*$-chain satisfies property (K), and the algorithm described for the width-2 case will then partition a $*$-chain into five chains in **P**. The theorem follows since $(5^n - 1)/4 = 1 + 5(5^{n-1} - 1)/4$. □

It is apparently a very difficult problem to determine just how good the upper bound in Theorem 3.2 actually is. For $n = 2$, Theorem 3.2 gives an upper bound of 6, but Felsner (1995) has just shown that the correct answer is 5. Saks observed that the techniques used to show that Theorem 3.1 is best possible can be dualized to produce a lower bound of the form $n(n+1)/2$ for Theorem 3.2. This bound is probably a very weak result, and it would be interesting to determine whether there is an algorithm which will partition on-line a poset of width at most n into n^c chains for some absolute constant c.

Recently, Kierstead et al. (1994) have shown that for every radius-two tree T, there exists a function $f_T : \mathbb{N} \to \mathbb{N}$, so that if G is any graph which does not contain T as an induced subgraph and does not contain a complete subgraph on k vertices, then G can be colored on-line with $f_T(k)$ colors. The complement of a comparability graph does not contain the subdivision of $K_{1,3}$, so it follows that there exists a function $g : \mathbb{N} \to \mathbb{N}$ so that a comparability graph with independence number n can be partitioned on-line into $g(n)$ complete subgraphs. This result does not follow from a straightforward extension of the ideas presented in this section. The difficulty is that the argument presented here makes specific use of the order relation between points–not just the information as to which pairs of points are comparable.

4. Sperner's lemma and the cross cut conjecture

A poset $\mathbf{P}$ is said to be *ranked* if every maximal chain in $\mathbf{P}$ has the same number of points. When $\mathbf{P}$ is ranked and $x \in X$, we let $r(x)$ be the largest i so that there exists a chain of i points having x as its least element. The value $r(x)$ is called the *rank* of x, and the antichains $A_i = \{x\colon r(x) = i\}$ are called *ranks*. The poset $\mathbf{P}$ is said to be a *Sperner* poset if the width of $\mathbf{P}$ equals the maximum cardinality of its ranks. The following now classic result is due to Sperner (1928).

Theorem 4.1. *For each $n \geqslant 1$, $\mathbf{2}^n$ is a Sperner poset. In particular, the width of $\mathbf{2}^n$ is $\binom{n}{\lfloor n/2 \rfloor}$ (cf. chapter 24).*

Proof. We consider $\mathbf{2}^n$ as the set of all subsets of $\{1, 2, \ldots, n\}$ ordered by inclusion. It is easy to see that the maximum cardinality of a rank of $\mathbf{2}^n$ is the binomial coefficient $\binom{n}{\lfloor n/2 \rfloor}$. Also there are $n!$ maximal chains in $\mathbf{2}^n$. Now suppose the width of $\mathbf{2}^n$ is t, and let $\mathscr{A} = \{A_1, A_2, \ldots, A_t\}$ be a maximum antichain in $\mathbf{2}^n$. If $A \in \mathscr{A}$ and $|A| = k$, then there are $k!(n-k)!$ maximal chains in $\mathbf{2}^n$ which contain A. It follows that $\sum_{i=1}^{t} k_i!(n-k_i)! \leqslant n!$ where $k_i = |A_i|$. Thus $t/\binom{n}{\lfloor n/2 \rfloor} \leqslant \sum_{i=1}^{t} k_i!(n-k_i)!/n! \leqslant 1$, so that $t \leqslant \binom{n}{\lfloor n/2 \rfloor}$ as claimed. □

An enormous amount of research has been done on generalizations of this elementary result, and we encourage the reader to consult chapter 24 or the book Anderson (1987) which concentrates on this subject. Also, chapter 32 contains an important result from Sperner theory. In view of our space limitations, we include here only an outline of the theorem of Canfield (1978) which asserts that sufficiently large partition lattices are not Sperner posets. The argument we give is patterned after the argument given subsequently by Shearer (1979).

The *partition lattice* $\mathbf{\Pi}_n$ is the poset whose elements are the partitions (into equivalence classes) of the set $\{1, 2, \ldots, n\}$. In $\mathbf{\Pi}_n$, we set $\pi_1 \leqslant \pi_2 \iff$ each class in π_1 is a subset of a class in π_2. Partition lattices are natural combinatorial objects and have been studied extensively both in combinatorial mathematics and in related areas. For example, an important theorem in lattice theory due to Pudlak and Tuma (1980) asserts that every lattice is a sublattice of a partition lattice. The problem of investigating the Sperner property for partition lattices was popularized by Rota.

The rank sizes of the partition lattice $\mathbf{\Pi}_n$ are the Stirling numbers of the second kind $S(n,k)$ for $k = 0, 1, 2, \ldots, n$. These numbers form a unimodal sequence achieving maximum value when $k = k_n \sim n/\log n$. If $\pi \in \mathbf{\Pi}_n$ and π has k_i classes of size i for each i, then π is called a partition of type $1^{k_1}2^{k_2}3^{k_3}\ldots n^{k_n}$.

The following lemma is an easy exercise.

Lemma 4.2. *The numbe*

$n!/(1!)^{k_1}(2!)^{k_2}\cdot\cdot$

Here is Canfield's (197

Theorem 4.3. *If n is su*
poset.

Proof. We actually prov
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forward extension to ob
type $m^{h_1}(2m)^{h_2}(3m)^{h_3}$ wh
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h_1, h_2, and h_3 satisfy:

$h_1 = [5(k+1) -$

$h_2 = [n/m - (k$

$h_3 = [n/m - (k$

The value θ is taken fro
the time being m is unsp
Any partition into k cl
to one of the following s

Type 1: m^{h_1-2}

Type 2: m^{h_1-1}

Type 3: m^{h_1-1}

Type 4: $m^{h_1}(2$

Type 5: $m^{h_1}(2$

Type 6: $m^{h_1}(2$

We will show that for su
these six types. The rema
in $\mathscr{P}$ will then form an a
Π_n is not a Sperner pose
By Lemma 4.2, the ra
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tedious) calculation to sh
as n tends to infinity. An
ratio of the number of pa
goes to 0 for each of the
all six ratios are sufficie
follows. □

Lemma 4.2. *The number of partitions in $\mathbf{\Pi}_n$ of type $1^{k_1}2^{k_2}\cdots n^{k_n}$ is*

$$n!/(1!)^{k_1}(2!)^{k_2}\cdots(n!)^{k_n}k_1!k_2!\cdots k_n!.$$

Here is Canfield's (1978) solution to Rota's conjecture.

Theorem 4.3. *If n is sufficiently large, the partition lattice $\mathbf{\Pi}_n$ is not a Sperner poset.*

Proof. We actually prove a slightly weaker result. We show that for certain large values of n, the partition lattice $\mathbf{\Pi}_n$ is not a Sperner poset. It is a relatively straightforward extension to obtain the general result. Let $\mathcal{P}$ consist of all partitions of type $m^{h_1}(2m)^{h_2}(3m)^{h_3}$ where $h_1+h_2+h_3=k+1$, and $k=k_n$ is chosen to maximize the Stirling number $S(n,k)$. Note that $mh_1+2mh_2+3mh_3=n$. Furthermore, h_1, h_2, and h_3 satisfy:

$$\begin{aligned} h_1 &= [5(k+1)-2n/m-\theta]/3, \\ h_2 &= [n/m-(k+1)+2\theta]/3, \\ h_3 &= [n/m-(k+1)-\theta]/3. \end{aligned}$$

The value θ is taken from $\{-1,0,+1\}$ so that each h_i is an integer. However, for the time being m is unspecified, although we assume that m divides n.

Any partition into k classes which is comparable to a partition in $\mathcal{P}$ must belong to one of the following six types:

Type 1: $m^{h_1-2}(2m)^{h_2+1}(3m)^{h_3}$

Type 2: $m^{h_1-1}(2m)^{h_2-1}(3m)^{h_3+1}$

Type 3: $m^{h_1-1}(2m)^{h_2}(3m)^{h_3-1}(4m)^1$

Type 4: $m^{h_1}(2m)^{h_2-2}(3m)^{h_3}(4m)^1$

Type 5: $m^{h_1}(2m)^{h_2-1}(3m)^{h_3-1}(5m)^1$

Type 6: $m^{h_1}(2m)^{h_2}(3m)^{h_3-2}(6m)^1$

We will show that for sufficiently large n, there are fewer than $|\mathcal{P}|$ partitions of these six types. The remaining partitions into k classes together with the partitions in $\mathcal{P}$ will then form an antichain of more than $S(n,k)$ elements which shows that Π_n is not a Sperner poset.

By Lemma 4.2, the ratio of the number of partitions of Type 1 divided by the number of partitions in $\mathcal{P}$ is $h_1(h_1-1)/(h_2+1)\binom{2m}{m}$. Then it is an easy (although tedious) calculation to show that if we set $m=\lfloor n/1.06k \rfloor$, then this ratio goes to 0 as n tends to infinity. An analogous computation shows that for this value of m, the ratio of the number of partitions of Type i divided by the number of partitions in $\mathcal{P}$ goes to 0 for each of the other five types. It may be shown that when $n > 4\times 10^9$, all six ratios are sufficiently small that their sum is less than 1 and the theorem follows. □

Some progress has been made in reducing the value of n for which it can be shown that $\mathbf{\Pi}_n$ is not a Sperner poset. Jichang and Kleitman (1984) have lowered the estimate to 3.4×10^6. However, the enormity of these estimates and the width of the corresponding partition lattices are striking testimony to the adage well known to researchers in combinatorical mathematics: Woe be to those who make conclusions based on detailed examinations of small examples. Sometimes we are startled to learn just how large small can be.

5. Linear extensions and correlation

Let $\mathbf{P}$ be a poset, and let $\mathscr{E}$ denote the set of all linear extensions of $\mathbf{P}$. It is natural to consider the elements of $\mathscr{E}$ as equally likely outcomes in a finite probability space. When x and y are distinct points in $\mathbf{P}$, we let $\mathrm{Prob}[x < y]$ denote the probability of the event consisting of all linear extensions in which $x < y$. Observe that $\mathrm{Prob}[x < y]$ is the ratio of the number of linear extensions with $x < y$ divided by $|\mathscr{E}|$. Note that $\mathrm{Prob}[x < y] = 1 \iff x < y$ in $\mathbf{P}$, and $\mathrm{Prob}[x < y] = 0 \iff y < x$ in $\mathbf{P}$.

Similarly, if x, y, z are three distinct points in $\mathbf{P}$, we let $\mathrm{Prob}[x < y | x < z]$ denote the conditional probability that in a random selection of a linear extension, the relation $x < y$ holds given that $x < z$ holds. In 1980, Rival and Sands made the following conjecture, which quickly became known as the xyz-conjecture.

Conjecture 5.1. If x, y, z are distinct points in a poset $\mathbf{P}$, then

$$\mathrm{Prob}[x < y] \leqslant \mathrm{Prob}[x < y | x < z].$$

It is easy to see that the xyz-conjecture holds except possibly when $\{x, y, z\}$ is a three-element antichain. In this case, Rival and Sands conjectured that the inequality in Conjecture 5.1 was strict, and this stronger version became known as the strict xyz-conjecture. The original conjecture was settled in the affirmative by Shepp (1980, 1982) using the FKG-inequality from statistical mechanics. The strict xyz-conjecture was proved by Fishburn (1984) using an important generalization of the FKG-inequality proved by Ahlswede and Daykin (1978), which we call the AD-inequality.

Although we do not include its proof here, the AD-inequality is a marvelous device with a growing list of applications in combinatorics. We encourage the reader to study its elementary proof carefully.

Let $\mathbf{P}$ be a distributive lattice with meet and join denoted $\wedge$ and $\vee$ respectively. When A and B are subsets of X, we let $A \wedge B = \{a \wedge b\colon a \in A \text{ and } b \in B\}$ and $A \vee B = \{a \vee b\colon a \in A \text{ and } b \in B\}$. When $f\colon X \to \mathbb{R}$ and $A \subset X$, let $f(A) = \sum_{x \in A} f(x)$. Let $\mathbb{R}_0$ denote the nonnegative real numbers.

Here is the AD-inequality.

Theorem 5.2. *Let $\mathbf{P}$ be a distributive lattice and let α, β, γ, and δ be functions from X to $\mathbb{R}_0$ satisfying:*

(i) $\alpha(x)\beta(y) \leqslant \gamma(x \wedge y)\delta(x \vee y)$ *for all* $x, y \in X$.
Then the following inequality holds for every $A, B \subset X$*:*
(ii) $\alpha(A)\beta(B) \leqslant \gamma(A \wedge B)\delta(A \vee B)$.

Corollary 5.3 (Fortuin et al. 1971). *Let* **P** *be a distributive lattice and let* $\mu : X \to [0, 1]$ *satisfy:*
(i) $\mu(x)\mu(y) \leqslant \mu(x \wedge y)\mu(x \vee y)$, *for all* $x, y \in X$.
If f *and* g *are monotonic functions from* X *to* $\mathbb{R}$, *then the following inequality holds:*
(ii) $[(f\mu)(X)][(g\mu)(X)] \leqslant [\mu(X)][(fg\mu)(X)]$.

Proof. Assume first that f and g map X to $\mathbb{R}_0$. For each $x \in X$, define $\alpha(x) = f(x)\mu(x)$, $\beta(x) = g(x)\mu(x)$, $\gamma(x) = \mu(x)$, and $\delta(x) = f(x)g(x)\mu(x)$. Then for an arbitrary $x, y, \in X$ we have:

$$\begin{aligned}\alpha(x)\beta(y) &= f(x)\mu(x)g(y)\mu(y)\\ &= f(x)g(y)\mu(x)\mu(y)\\ &\leqslant f(x)g(y)\mu(x \wedge y)\mu(x \vee y)\\ &\leqslant \mu(x \wedge y)f(x \vee y)g(x \vee y)\mu(x \vee y)\\ &= \gamma(x \wedge y)\delta(x \vee y).\end{aligned}$$

Therefore $\alpha(X)\beta(X) \leqslant \gamma(X)\delta(X)$ as claimed. When the range of f or g includes negative reals, we increase both functions by some suitably large constant. □

A subset S of a poset **P** is called a *down set* (*up set*) if $x \in S$ and $y \leqslant x$ in **P** ($x \leqslant y$ in **P**) always implies $y \in S$.

Corollary 5.4. *Let* U_1 *and* U_2 *be up sets in a distributive lattice* **P**. *Then* $|U_1||U_2| \leqslant |U_1 \cap U_2||X|$.

Proof. Set $\alpha(x) = \beta(x) = \gamma(x) = \delta(x) = 1$ for every $x \in X$. Observe that $U_1 \vee U_2 = U_1 \cap U_2$ and $U_1 \wedge U_2 \subset X$. □

The special case of Corollary 5.4 when $\mathbf{P} = \mathbf{2}^n$ was first proved by Kleitman (1966) and in dual form by Seymour (1973).

We close this section by outlining Shepp's (1982) proof of the xyz-conjecture.

Theorem 5.5. *Let* **P** *be a poset and let* x, y, *and* z *be three distinct points in* X. *Then*

$$\text{Prob}[x < y] \leqslant \text{Prob}[x < y | x < z].$$

Proof. We assume $\{x, y, z\}$ is a three-element antichain in **P**. Let k be a positive integer, and let Y_k denote the set of all order preserving functions from **P** to **k**. Define a partial order P_k on Y_k by setting $f \leqslant g$ in $P_k \Longleftrightarrow f(x) \geqslant g(x)$ and

$f(u) - f(x) \leqslant g(u) - g(x)$ for every $u \in X$. It is straightforward to verify that the poset (Y_k, P_k) is in fact a distributive lattice.

Now let $U_1(k) = \{f \in Y_k\colon f(x) < f(y)\}$ and $U_2(k) = \{f \in Y_k\colon f(x) < f(z)\}$. Then $U_1(k)$ and $U_2(k)$ are up sets in the distributive lattice (Y_k, P_k). Therefore

$$|U_1(k)||U_2(k)| \leqslant |U_1(k) \cap U_2(k)||Y_k|,$$

so that:

$$\frac{|U_1(k)|}{|Y_k|} \leqslant \frac{|U_1(k) \cap U_2(k)|/|Y_k|}{|U_2(k)|/|Y_k|}.$$

However, it is easy to see that as k tends to ∞, the left-hand side of this inequality approaches $\mathrm{Prob}[x < y]$ while the right-hand side approaches $\mathrm{Prob}[x < y | x < z]$. □

The reader should note that the truly clever part of this proof is the nonstandard definition of the partial order P_k so that (Y_k, P_k) is a distributive lattice having $U_1(k)$ and $U_2(k)$ as up sets. Fishburn's (1984) proof of the strict xyz-inequality requires two applications of the AD-inequality. Winkler's (1986) survey article is a good starting point for an overview of work on correlation.

6. Balancing pairs and the $\frac{1}{3}$–$\frac{2}{3}$ conjecture

The following conjecture is due to Kislitysn (1968), although it was also made independently by Fredman (1979) and Linial (1984):

Conjecture 6.1 (*The $\frac{1}{3}$–$\frac{2}{3}$ conjecture*). If $\mathbf{P}$ is not a chain, then $\mathbf{P}$ contains distinct points x and y for which $\frac{1}{3} \leqslant \mathrm{Prob}[x > y] \leqslant \frac{2}{3}$.

If true, this conjecture is best possible as is evidenced by the three-point poset $\mathbf{2}+\mathbf{1}$. Given a poset $\mathbf{P} = (X, P)$ which is not a chain, let $\delta(\mathbf{P})$ denote the largest positive number for which there exists a pair $x, y \in X$ with $\delta(\mathbf{P}) \leqslant \mathrm{Prob}[x > y] \leqslant 1 - \delta(\mathbf{P})$. Using this terminology, we can restate the $\frac{1}{3}$–$\frac{2}{3}$ conjecture as follows.

Conjecture 6.2. If $\mathbf{P}$ is not a chain, then $\delta(\mathbf{P}) \geqslant \frac{1}{3}$.

The original motivation for studying balancing pairs in posets was the connection with sorting. The problem was to answer whether it is always possible to determine an unknown linear extension of a poset $\mathbf{P}$ with $O(\log t)$ rounds (questions) where t is the number of linear extensions of P. The answer would be "yes" if one could prove that there exists an absolute constant δ_0 so that $\delta(\mathbf{P}) \geqslant \delta_0$ for any $\mathbf{P}$ which is not a chain.

Linial (1984) has shown that the $\frac{1}{3}$–$\frac{2}{3}$ conjecture holds for posets of width two. Fishburn et al. (1992) show that it holds for posets of height at most two. Although the conjecture remains open for general posets, we present a partial result, due to Kahn and Saks (1984), which is particularly appealing in view of its nontrivial application of the Alexandrov–Fenchel inequalities for mixed volumes.

Theorem 6.3. *If* **P** *is not a chain, then* X *contains a distinct pair* x, y *so that*

$$\tfrac{3}{11} \leqslant \text{Prob}[x > y] \leqslant \tfrac{8}{11},$$

and thus $\delta(\mathbf{P}) \geqslant \frac{3}{11}$.

Proof. Clearly, we may assume that **P** does not have a least element. Let $\mathscr{E}$ denote the set of all linear extensions of **P** and let $n = |X|$. For each $L \in \mathscr{E}$, let $h_L : X \to \mathbf{n}$ be the order preserving injection determined by L. Let $|\mathscr{E}| = t$ and then define $h : X \to \mathbb{R}_0$ by $h(x) = (\sum_{L\in\mathscr{E}} h_L(x))/t$. The value $h(x)$ is the average height of x among the linear extensions in $\mathscr{E}$. Since no element satisfies $h(x) = 0$, it follows that there exist distinct elements $x, y \in X$ with $0 \leqslant h(y) - h(x) < 1$. Note that such a pair must be incomparable. We will show that $\frac{3}{11} \leqslant \text{Prob}[x > y] \leqslant \frac{8}{11}$. The argument depends on a series of lemmas which hold for an arbitrary incomparable pair in **P**.

Fix an arbitrary incomparable pair $x, y \in X$. For a positive (negative) integer i, let e_i be the number of linear extensions of **P** in which x is below (above) y by exactly i positions.

Lemma 1. $e_1 = e_{-1}$.

Proof. Since x and y occur consecutively, they may be interchanged. □

Lemma 2. $e_2 + e_{-2} \leqslant e_1 + e_{-1}$.

Proof. Suppose $L \in \mathscr{E}$ and $|h_L(y) - h_L(x)| = 2$. Let u be the unique point between x and y. If $u \parallel x$, exchange u and x. If u is comparable with x, then $u \parallel y$. In this case, the cyclic permutation (uyx) converts L into an extension in which x and y are consecutive. The mapping is easily seen to be an injection. □

Lemma 3. *If* $|i| \geqslant 2$, *then* $e_i \leqslant e_{i-1} + e_{i+1}$.

Proof. Without loss of generality $i \geqslant 2$. Suppose the lemma is false and of all counterexamples, choose one for which $|X| = n$ is minimum. For this value of n, we choose **P** so that the number of comparable pairs is maximum. We first show that for every incomparable pair $u, v \in X$, one of u and v is greater than or equal to x and the other is less than or equal to y.

Suppose to the contrary that $u \parallel v$ and that the above statement does not hold. Let $\mathbf{P}'$ be the poset obtained by adding the relation $u < v$ to the partial order on X and taking the transitive closure. Also let $\mathbf{P}''$ be the poset obtained by adding $v < u$. In $\mathbf{P}'$, we let e'_j denote the number of linear extensions with x below y by exactly j positions. Also, let e''_j denote the corresponding number for $\mathbf{P}''$. Then $e_i = e'_j + e''_j$ for each $j \geqslant 1$. Since $\mathbf{P}'$ and $\mathbf{P}''$ have more comparable pairs than **P**, we know that $e'_i \leqslant e'_{i-1} + e'_{i+1}$ and $e''_i \leqslant e''_{i-1} + e''_{i+1}$, so that $e_i \leqslant e_{i-1} + e_{i+1}$ as claimed.

We next claim that $x \in \text{MIN}(\mathbf{P})$ and $y \in \text{MAX}(\mathbf{P})$. For suppose $u \in \text{MIN}(\mathbf{P})$ and that $u < x$. If $u < y$ in **P**, then n must be comparable with every point in X, i.e.,

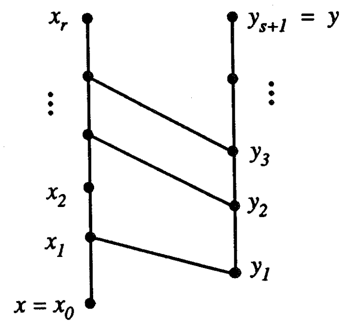

Figure 6.1.

u is the least element of $\mathbf{P}$. In this case, we consider the poset $\mathbf{P}' = \mathbf{P} - \{u\}$ and the numbers e'_j defined in the natural way for $\mathbf{P}'$. Since $e_j = e'_j$, we would conclude $e_i \leqslant e_{i-1} + e_{i+1}$. The contradiction shows $x \in \mathrm{MIN}(\mathbf{P})$. Dually, $y \in \mathrm{MAX}(\mathbf{P})$.

These remarks show that X is the union of two chains $x = x_0 < x_1 < x_2 < \cdots < x_r$ and $y_1 < y_2 < \cdots < y_s < y_{s+1} = y$. Furthermore, no y_i is larger than any x_j in $\mathbf{P}$. (See fig. 6.1.)

We now distinguish two cases.

Case 1. $i < n-1$. Choose $L \in \mathscr{E}$ with $h_L(y) - h_L(x) = k$. If x is not the least element in L, exchange x with the element immediately under it. If x is the least element in L, then y is not the greatest. Let z be the element immediately over x. If $x \parallel z$, exchange them; otherwise exchange y with the element immediately over it. This procedure is an injection which transforms L into a linear extension in which x is below y by either $i-1$ or $i+1$ positions.

Case 2. $i = n-1$. For each subposet $\mathbf{Q} \subset \mathbf{P}$, let $e(\mathbf{Q})$ denote the number of linear extensions of $\mathbf{Q}$. Let $\mathbf{Z} = \mathbf{P} - \{x, y\}$. Then $e_{i-1} = e(\mathbf{Z} - \{x_r\}) + e(\mathbf{Z} - \{y_1\})$, $e_i = e(\mathbf{Z})$ and $e_{i+1} = 0$. Now $e(\mathbf{Z}) = e(\mathbf{Z} - \{x_1\}) + e(\mathbf{Z} - \{y_1\})$ so, to complete the argument, we need only show that $e(\mathbf{Z} - \{x_r\}) \geqslant e(\mathbf{Z} - \{x_1\})$. However, this inequality follows immediately since the mapping $x_j \to x_{j-1}$ transforms a linear extension of $\mathbf{Z} - \{x_1\}$ into a linear extension of $\mathbf{Z} - \{x_r\}$. This completes the proof of Lemma 3. □

The next lemma is a special case of the Alexandrov–Fenchel inequalities for mixed volumes. This method was pioneered by Stanley (1981).

Lemma 4. *Let K_0 and K_1 be convex subsets of $\mathbb{R}^n$. For each λ with $0 < \lambda < 1$, let $K_\lambda = \{(1-\lambda)v_0 + \lambda v_1 \colon v_0 \in K_0,\ v_1 \in K_1\}$. Let d be the dimension of the affine hull*

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of K_λ for $\lambda \in (0,1)$. Then there exist unique numbers $a_0, a_1, \ldots, a_d$ so that for all λ, the d-dimensional volume of K_λ satisfies:

$$\mathrm{Vol}(K_\lambda) = \sum_{k=0}^{d} \binom{d}{k} a_k (1-\lambda)^{d-k} \lambda^k.$$

Furthermore, the sequence $a_0, a_1, a_2, \ldots, a_d$ is logarithmically concave, i.e.,

$$a_k^2 \geqslant a_{k-1} a_{k+1}, \quad \text{for } k = 1, 2, \ldots, d-1.$$

Lemma 5. *The sequences $e_1, e_2, e_3, \ldots, e_n$ and $e_{-n}, e_{-n+1}, \ldots, e_{-1}$ are logarithmically concave.*

Proof. Let $\mathbb{R}^{\mathbf{P}}$ denote the vector space of all order preserving functions from $\mathbf{P}$ to $\mathbb{R}$. Also let $C(\mathbf{P}) = \{f \in \mathbb{R}^{\mathbf{P}}\colon 0 \leqslant f(x) \leqslant 1 \text{ for all } x \in \mathbf{P}\}$. For λ with $0 \leqslant \lambda \leqslant 1$, let $K_\lambda = \{f \in C(\mathbf{P})\colon f(y) - f(x) = \lambda\}$. Note that $K_\lambda = (1-\lambda)K_0 + \lambda K_1$. Also note that when $0 < \lambda < 1$, K_λ has dimension $n-1$.

For each $L \in \mathscr{E}$ with $x < y$ in L, let $\Delta_\lambda(L) = \{f \in K_\lambda\colon v < w \text{ in } L \Rightarrow f(v) \leqslant f(w)\}$. Then K_λ is the union of the $\Delta_\lambda(L)$'s taken over all L with $x < y$ in L. Since the $\Delta_\lambda(L)$'s have disjoint interiors, we see that

$$\mathrm{Vol}(K_\lambda) = \sum_{\substack{L \in \mathscr{E} \\ x<y \text{ in } L}} \mathrm{Vol}(\Delta_\lambda(L)).$$

Now consider some particular $L \in \mathscr{E}$ with $x < y$ in L. Suppose L orders X as $v_0 < v_1 < \cdots < v_{n-1}$ with $x = v_i$ and $y = v_j$. Then $\Delta_\lambda(L)$ consists of those $f \in C(\mathbf{P})$ for which $0 \leqslant f(v_0) \leqslant f(v_1) \leqslant \cdots \leqslant f(v_{n-1}) \leqslant 1$ and $f(v_j) = f(v_i) + \lambda$. Now consider the volume preserving mapping defined by:

$$\begin{aligned} f(v_k) &\to f(v_k) - \lambda && \text{when } k > j; \\ f(v_k) &\to f(v_k) - f(v_i) && \text{when } j \geqslant k > i; \quad \text{and} \\ f(v_k) &\to f(v_k) && \text{when } i \geqslant k. \end{aligned}$$

Then the image of $\Delta_\lambda(L)$ under this mapping is the set of $f \in C(\mathbf{P})$ for which

$$0 \leqslant f(v_0) \leqslant f(v_1) \leqslant \cdots \leqslant f(v_i) \leqslant f(v_{j+1}) \leqslant \cdots \leqslant f(v_{n-1}) \leqslant 1 - \lambda$$

and $0 \leqslant f(v_i) \leqslant f(v_{i+1}) \leqslant \cdots \leqslant f(v_j) = \lambda$. However, this set is the product of two simplices and its volume is therefore

$$\frac{(1-\lambda)^{n-j+1}}{(n-j+i)!} \; \frac{\lambda^{j-i-1}}{(j-i-1)!}.$$

It follows that

$$\begin{aligned} \mathrm{Vol}(K_\lambda) &= \sum_{0 \leqslant i < j \leqslant n-1} \sum_{\substack{L \in \mathscr{E} \\ h_L(x)=i \\ h_L(y)=j}} \frac{(1-\lambda)^{n-j+i}}{(n-j+i)!} \; \frac{\lambda^{j-i-1}}{(j-i-1)!} \\ &= \sum_{i \geqslant 1} e_i \frac{(1-\lambda)^{n-i}}{(n-i)!} \; \frac{\lambda^{i-1}}{(i-1)!}. \end{aligned}$$

From Lemma 4, we conclude that $a_i = e_{i-1}/(n-1)!$, and thus the sequence $e_1, e_2, e_3, \ldots$ is logarithmically concave. The argument for the other sequence is dual. $\square$

For the remainder of the proof, we assume that x and y are an incomparable pair with $0 \leqslant h(y) - h(x) \leqslant 1$. For each $i \geqslant 1$, we let $b_i = e_i/t$ and $a_i = e_{-i}/t$. We then know that the following conditions hold:

(1) $a_1, a_2, a_3, \ldots$ and $b_1, b_2, b_3, \ldots$ are sequences of nonnegative real numbers so that $\sum_{i\geqslant 1} a_i + \sum_{i\geqslant 1} b_i = 1$.
(2) $a_1 = b_1$.
(3) $a_2 + b_2 \leqslant a_1 + b_1$.
(4) $a_i \leqslant a_{i-1} + a_{i+1}$ and $b_i \leqslant b_{i-1} + b_{i+1}$ for all $i \geqslant 2$.
(5) $a_i^2 \geqslant a_{i-1}a_{i+1}$ and $b_i^2 \geqslant b_{i-1}b_{i+1}$ for all $i \geqslant 2$.
(6) If $a_i = 0$, then $a_{i+1} = 0$, for all $i \geqslant 1$.
(7) If $b_i = 0$, then $b_{i+1} = 0$, for all $i \geqslant 1$.

It remains only to show that whenever these seven properties are satisfied, we always have $\sum_{i\geqslant 1} ia_i - \sum_{i\geqslant 1} ib_i \geqslant 1$ whenever $\sum_{i\geqslant 1} b_i \leqslant \frac{3}{11}$. To accomplish this, we fix a value $b = b_1 = a_1$, and the value $B = \sum_{i\geqslant 1} b_i$. For the pair (b, B), we determine the unique sequences $\{a_i\colon i \geqslant 1\}$, $\{b_i\colon i \geqslant 1\}$ satisfying these seven conditions which minimize $\sum ia_i - \sum ib_i$.

The infinite geometric sequence obtained by setting $\varepsilon = b/B$ and $b_i = b(1-\varepsilon)^{i-1}$ is easily seen to maximize $\sum ib_i$ among all sequences $\{b_i\colon i \geqslant 1\}$ satisfying $b_1 = b$, $\sum b_i = B$, and $b_i^2 \geqslant b_{i-1}b_{i+1}$ for all $i \geqslant 2$. The argument to minimize $\sum ia_i$ is somewhat more complicated. We know $a_2 + b_2 \leqslant a_1 + b_1 = 2b$ so $a_2 \leqslant 2b - b_2 = b(1+\varepsilon)$. Therefore $a_{i+1} \leqslant (1+\varepsilon)a_i$ for every $i \geqslant 1$. On the other hand, set $A = 1 - B$ and for each $j \geqslant 1$, let $s_j = A - \sum_{i=1}^{j} a_i$. Then $a_{j+1} \leqslant s_j$. Also $a_{j+1} \leqslant a_j + a_{j+2} \leqslant a_j + s_{j+1} = a_j + s_j - a_{j+1}$, so that $a_{j+1} \leqslant (a_j + s_j)/2$. It is then easy to verify that $\sum ia_i$ is minimized when there is some $k \geqslant 1$ so that $a_i = b(1+\varepsilon)^{i-1}$ for all $i = 1, 2, \ldots, k$ and either:

Type 1: $a_{k+1} = s_k$, $a_i = 0$ for all $i \geqslant k+2$; or

Type 2: $a_{k+1} = (s_k + a_k)/2$ where $s_k > a_k$, $a_{k+2} = s_{k+1} = (s_k - a_k)/2$, and $a_i = 0$ for all $i \geqslant k+3$.

Set $\alpha = a_{k+1}/a_k$ and $\beta = a_{k+2}/a_k$. We verify that $\sum ia_i - \sum ib_i \geqslant 1$ for a Type 2 sequence $\{a_i\colon i \geqslant 1\}$. The reader may enjoy the challenge of handling the Type 1 case – it is somewhat easier. Now for a Type 2 sequence, we know:

$$1 + \varepsilon \geqslant \alpha \geqslant 1, \quad \beta = \alpha - 1, \text{ and} \tag{1}$$

$$\alpha + \beta = \frac{(1+\varepsilon)^{1-k}}{\varepsilon B} - 1 - 1/\varepsilon. \tag{2}$$

Using the definitions of $\{a_i\colon i \geqslant 1\}$ and $\{b_i\colon i \geqslant 1\}$, we find that

$$\sum ia_i - \sum ib_i = \frac{[k\varepsilon - \varepsilon - 1 + \varepsilon^2 k(\alpha + \beta + 1) + \varepsilon^2(\alpha + 2\beta)]}{\varepsilon + \varepsilon^2(\alpha + \beta + 1)}.$$

The inequality $\sum ia_i - \sum ib_i \geqslant 1$ is then equivalent (using (1)) to

$$\frac{(1+\varepsilon)^{k+1}}{k\varepsilon} \leqslant \frac{1}{B} + \frac{\varepsilon\beta}{k}(1+\varepsilon)^{k-1}. \tag{3}$$

This in turn is equivalent to

$$\frac{(4\varepsilon^2+5\varepsilon+2)(1+\varepsilon)^{k-1}}{(2k+1)\varepsilon} \leqslant \frac{1}{\beta}. \tag{4}$$

Now $\beta = \frac{1}{2B\varepsilon(1+\varepsilon)^{k-1}} - \frac{1}{2\varepsilon} - 1$ so the inequality $0 \leqslant \beta \leqslant \varepsilon$ converts to

$$(2\varepsilon+1)(1+\varepsilon)^{k-1} \leqslant \frac{1}{B} \tag{5}$$

and

$$(2\varepsilon^2+2\varepsilon+1)(1+\varepsilon)^{k-1} \geqslant \frac{1}{B}. \tag{6}$$

We may assume $\varepsilon < 2/(2k-1)$; otherwise $k \geqslant 1/\varepsilon + \frac{1}{2}$ and

$$\begin{aligned}\frac{(4\varepsilon^2+5\varepsilon+2)(1+\varepsilon)^{k-1}}{(2k+1)\varepsilon} &\leqslant \frac{(4\varepsilon^2+5\varepsilon+2)(1+\varepsilon)^{k-1}}{2+2\varepsilon}\\ &\leqslant (2\varepsilon+1)(1+\varepsilon)^{k-1}\\ &\leqslant 1/B.\end{aligned}$$

We may also assume $\varepsilon > 3/(3k+1)$, for if $\varepsilon \leqslant 3/(3k+1)$, inequality (5) implies

$$\left(1+\frac{3}{3k+1}\right)^{k-1}\left[\frac{18}{(3k+1)^2} + \frac{6}{3k+1} + 1\right] \geqslant \frac{11}{3}. \tag{7}$$

However, inequality (6) is false for all $k \geqslant 1$.

To complete the proof, we observe that the left-hand side of inequality (3) is an increasing function of ε when $\varepsilon > 3/(3k+1)$. It suffices to test the validity of (3) at an upper bound for ε. When $k = 1$, the trivial bound $\varepsilon \leqslant 1$ works. For $k \geqslant 2$, use the bound $\varepsilon \leqslant 2/(2k-1)$. This completes the proof. □

In a certain sense, the approach taken by Kahn and Saks cannot be improved. In fig. 6.2, we show a poset **P** containing two points x, y satisfying $h(y) - h(x) = 1$ and $\text{Prob}[x > y] = \frac{3}{11}$.

Other proofs bounding $\delta(\mathbf{P})$ away from zero have been given. Khachiyan (1989) uses geometric techniques to show $\delta(\mathbf{P}) \geqslant 1/e^2$. Kahn and Linial (1991) provide a short and elegant argument using the Brunn–Minkowski theorem to show that $\delta(\mathbf{P}) \geqslant 1/2e$. Friedman (1993) also applies geometric techniques to obtain even better constants when the poset satisfies certain additional properties. Kahn and Saks (1984) conjectured that $\delta(\mathbf{P})$ approaches $\frac{1}{2}$ as the width of **P** tends to infinity. Komlós (1990) provides support for this conjecture by showing that for every

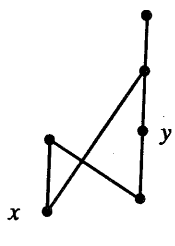

Figure 6.2.

$\varepsilon > 0$, there exists a function $f_\varepsilon(n) = \mathrm{o}(n)$ so that if $\mathbf{P} = (X, P)$ is a poset with $|X| = n$ and at least $f_\varepsilon(n)$ minimal points, then $\delta(\mathbf{P}) > \frac{1}{2} - \varepsilon$.

As Kahn and Saks (1984) point out, the value of the constant in Theorem 6.3 could be improved if we could show that there exists a positive absolute constant γ so that if $\mathbf{P}$ is not a chain, then it is always possible to find an ordered pair (x, y) with $0 \leqslant h(y) - h(x) \leqslant 1 - \gamma$. However, nobody has yet been able to settle whether such a γ exists. If it does, then as shown by Saks (1985), it must satisfy $\gamma \leqslant 0.133$. Even this value would not be enough to prove $\delta(\mathbf{P}) \geqslant \frac{1}{3}$. However, it is of interest to determine the maximum value of $|h(y) - h(x)|$ which allows one to conclude that $\frac{1}{3} \leqslant \mathrm{Prob}[x > y] \leqslant \frac{2}{3}$. Felsner and Trotter (1993) discuss how to modify the Kahn–Saks proof technique to obtain the next result, which is clearly best possible.

Theorem 6.4. *Let (x, y) be distinct points in a poset $\mathbf{P}$, and suppose that $0 \leqslant h(y) - h(x) \leqslant \frac{2}{3}$. Then $\frac{1}{3} \leqslant \mathrm{Prob}[x > y] \leqslant \frac{2}{3}$.*

Felsner and Trotter (1993) obtain a slight improvement in the Kahn–Saks bound by considering subposets in which the points are relatively close in average height.

Theorem 6.5. *There exists an absolute constant $\varepsilon > 0$ so that if $\mathbf{P}$ is a poset which is not a chain, then $\delta(\mathbf{P}) > \varepsilon + \frac{3}{11}$.*

In developing this theorem, Felsner and Trotter made a correlation conjecture which is of independent interest. Let x, y and z be distinct points in a poset $\mathbf{P} = (X, P)$. For each $i, j \in \mathbb{Z}$, let $p(i, j)$ denote the probabilty that $h_L(y) - h_L(x) = i$ and $h_L(z) - h_L(y) = j$ in a random linear extension L of P. Felsner and Trotter then made the following *cross product* conjecture.

Conjecture 6.6. Let $x <: y <: z$ in a poset $\mathbf{P} = (X, P)$. Then for all $i, j \in \mathbb{N}$,

$$p(i, j)p(i + 1, j + 1) \leqslant p(i, j + 1)p(i + 1, j).$$

Brightwell, Felsner and Trotter (1995) prove the cross product conjecture in the special case $i = j = 1$. They then prove the following lower bound.

Theorem 6.7. *If* **P** *is a fi*

$$\delta(\mathbf{P}) > \frac{5 - \sqrt{5}}{10}.$$

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7. Dimension and posets

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Theorem 6.7. *If* **P** *is a finite poset which is not a chain then*

$$\delta(\mathbf{P}) > \frac{5-\sqrt{5}}{10}.$$

The results of this section have emphasized existence questions – disregarding the issue as to how one actually goes about finding an incomparable pair of points which are balanced. Brightwell and Winkler (1991) showed that the problem of computing the number of linear extensions of a poset is #P-complete. However, if one is willing to use randomized algorithms, then a good approximation to the volume of a polytope can be efficiently computed, so that the theorems presented here can form the basis for a sorting algorithm.

On the other hand, if we limit our attention to deterministic algorithms, then an alternative approach is necessary. Kahn and Kim (1994) use a concept of entropy for posets and show the existence of a polynomial-time deterministic algorithm for sorting in $O(\log t)$ rounds. Their algorithm shows how to efficiently locate pairs to use in queries so that, regardless of the responses, the determination of the unknown linear extension is made in $O(\log t)$ rounds. However, at individual rounds, the pairs need not be balanced in the sense that for a given pair (x, y) used in the algorithm, $\text{Prob}[x > y]$ may be arbitrarily close to zero.

7. Dimension and posets of bounded degree

The *dimension* of a poset $\mathbf{P} = (X, P)$ is the least t for which there exists a family $R = \{L_1, L_2, \ldots, L_t\}$ of linear extensions of P so that $P = L_1 \cap L_2 \cap \cdots \cap L_t$. In fact, the dimension of **P** is the least t for which there exists a family $R = \{L_1, L_2, \ldots, L_t\}$ of linear orders (not necessarily linear extensions of P) on X so that whenever $x \not\leqslant y$ in P, there exists at least one i with $y < \{z\colon x \leqslant z \text{ in } P\}$ in L_i. The concept of dimension was introduced by Dushnik and Miller (1941).

In section 2, we presented Dilworth's decomposition theorem, but we did not give much of an explanation for the role this theorem plays in research on posets – other than to motivate the Greene–Kleitman theorem. However, Dilworth's theorem plays a major role in dimension theory. For example, Hiraguchi (1951) used it to prove that the dimension of a poset never exceeds its width. Hiraguchi (1951) also proved that the dimension of a poset $\mathbf{P} = (X, P)$ is at most $|X|/2$, provided $|X| \geqslant 4$. On the other hand, it is still unknown whether every poset **P** with three or more points always contains a pair whose removal decreases the dimension at most one (see Kelly 1984, Reuter 1989, Kierstead and Trotter 1991).

Dilworth's theorem is critical to dimension-theoretic ineqalities appearing in Bogart and Trotter (1973), Trotter (1974b, 1975a,c, 1976a). It also plays a major role in the variants of dimension investigated in Bogart and Trotter (1976a,b), Kierstead et al. (1987), Kierstead and Trotter (1985, 1989). For additional background material on dimension, the reader is encouraged to consult the monograph Trotter (1992) and the survey articles Kelly and Trotter (1982) and Trotter (1982).

Also, discussions of open problems in dimension theory are given in Trotter (1989, 1992, 1994).

In the remainder of this section, we discuss one important problem for which there is an interesting partial solution – utilizing the probabilistic method on posets. Brightwell's survey article (Brightwell 1993) highlights the recent work in this rapidly growing area of research.

For integers $n \geqslant 3$, $k \geqslant 0$, define the *crown* $\mathbf{S}_n^k$ as the poset of height 2 having $n+k$ minimal elements $a_1, a_2, \ldots, a_{n+k}$ and $n+k$ maximal elements $b_1, b_2, \ldots, b_{n+k}$ with $a_i < b_{i-1}, b_{i-2}, b_{i-3}, \ldots, b_{i-n+1}$ for each i (cyclically). Trotter (1974a) showed that $\dim(\mathbf{S}_n^k) = \lceil 2(n+k)/(k+2) \rceil$. As noted previously, Hiraguchi (1951) proved that the dimension of a poset on m points does not exceed $m/2$, when $m \geqslant 4$, so the crowns $\{\mathbf{S}_n^0 \colon n \geqslant 3\}$ show that this inequality is best possible. In fact, the crown $\mathbf{S}_n^0$ is called the *standard* example of an n-dimensional poset. However, it is of interest to investigate conditions which force the dimension of a poset to be small in comparison to the number of points.

We now proceed to study one such condition which surfaced in the investigation of crowns. For a point x in a poset $\mathbf{P}$, let $\deg(x)$ count the number of points comparable (but not equal) to x in $\mathbf{P}$. Then let $\Delta(\mathbf{P})$ denote the maximum value of $\deg(x)$ for $x \in \mathbf{P}$. Rödl and Trotter proved that if $\Delta(\mathbf{P}) \leqslant k$, then $\dim(\mathbf{P}) \leqslant 2k^2 + 2$. For each $x \in \mathbf{P}$, let $U(x) = \{y \colon x < y \text{ in } \mathbf{P}\}$ and let $u = \max\{|U(x)| \colon x \in \mathbf{P}\}$. Then $u \leqslant \Delta(\mathbf{P})$.

We now present a strengthening of this result due to Füredi and Kahn (1986). Lemmas 7.1, 7.2, 7.5, Corollary 7.4 and Theorem 7.6 all come from that paper.

Lemma 7.1. $\dim(\mathbf{P}) < 2(u+1)(\log |X|) + 1$.

Proof. Let $|X| = n$ and set $t = \lceil 2(u+1)\log n \rceil$. Let $L_1, L_2, \ldots, L_t$ be linear orders of X chosen at random. Then for each $i = 1, 2, \ldots, t$ and for each $x \in \mathbf{P}$, the probability that $x < y$ in L_i for all $y \in U(x)$ is at least $1/(u+1)$. Hence the probability that no L_i satisfies $x < y$ in L_i for all $y \in U(x)$ is at most $[1 - 1/(u+1)]^t < 1/n^2$. Thus the probability that there exists a pair $x, y \in X$ with $y \not\leqslant x$ in X but there is no L_i with $x < y$ in L_i for all $y \in U(x)$ is less than 1. Thus $\dim(\mathbf{P}) \leqslant t$ as claimed. □

For integers n, k with $1 < k \leqslant n$, let $\dim(1,k;n)$ denote the dimension of the poset formed by the 1-element and k-element subsets of $\{1, 2, \ldots, n\}$ ordered by inclusion. Then $\dim(1,k;n)$ is the least t for which there exist t linear orders $L_1, L_2, \ldots, L_t$ on $\{1, 2, \ldots, n\}$ so that for each $(k+1)$-element subset $S \subset \{1, 2, \ldots, n\}$ and for each $x \in S$, there is at least one L_i in which x is the least element of S. The following lemma follows along the same lines as Lemma 7.1.

Lemma 7.2. *For all integers n, k with $1 < k \leqslant n$, $\dim(1,k;n) \leqslant k^2(1 + \log(n/k))$.*

The Füredi–Kahn argument also depends on the following result, which is known as the Lovász local lemma (Erdős and Lovász 1973). Other applications of this lemma are given in chapter 33 by Spencer.

Lemma 7.3. *Let G be a graph on $\{1,2,\ldots,m\}$ and let $k=\Delta(G)$ denote the maximum degree in G. Suppose $A_1,A_2,\ldots,A_m$ are events in a probability space so that for each $i=1,2,\ldots,m$:*
(1) $\mathrm{Prob}[A_i]\leqslant 1/4k$, *and*
(2) *A_i is jointly independent of the events $\{A_j$: ij is not an edge in $G\}$.*
Then $\mathrm{Prob}[\bar{A}_i\bar{A}_2\cdots\bar{A}_m]>0$.

Corollary 7.4. *Let $b\geqslant 500$, $s=\lceil b/\log b\rceil$, and $v=\lceil 4.7\log b\rceil$. Let $\mathcal{H}$ be a hypergraph whose edges are subsets of sizes at most b from a set X. Suppose further that no point of X belongs to more than b edges in $\mathcal{H}$. Then there is a partition $X=X_1\cup X_2\cup\cdots\cup X_s$ so that $|H\cap X_i|\leqslant v$ for every edge $H\in\mathcal{H}$ and every $i=1,2,\ldots,s$.*

Proof. Let $X=X_1\cup X_2\cup\cdots\cup X_s$ be a random partition of X. We denote by $A(H,i)$ the event $|H\cap X_i|>v$. Then let G be the graph whose vertex set consists of the pairs (H,i) where H is an edge in $\mathcal{H}$ and $1\leqslant i\leqslant s$. The edges of G are the pairs $(H,i)(H',i')$ for which $H\cap H'\neq\emptyset$. Therefore $\Delta(G)\leqslant(1+b(b-1))s\leqslant b^3$.

However, since $|H|\leqslant b$,

$$\begin{aligned}\mathrm{Prob}[A(H,i)]&=\mathrm{Prob}[|H\cap X_i|>v]\\&\leqslant\sum_{t>v}\binom{b}{t}\left(\frac{1}{s}\right)^t\left(1-\frac{1}{s}\right)^{b-t}\\&<\frac{1}{3}\binom{b}{v}\left(\frac{1}{s}\right)^v\left(1-\frac{1}{s}\right)^{b-v}\\&<\frac{1}{3}\left(\frac{be}{vs}\right)^v\frac{1}{\sqrt{2\pi v}}\left(1-\frac{1}{s}\right)^{b-v}\\&<\frac{1}{4b^3}.\end{aligned}$$

The conclusion then follows from the Lovász local lemma. □

We also need the following lemma whose elementary proof we leave as an exercise.

Lemma 7.5. *Let a and b be positive integers. Let $\mathcal{H}$ be a hypergraph on the vertex set Y so that each edge in $\mathcal{H}$ has at most a elements and no vertex of Y belongs to more than b edges. Then there exists a partition $Y=Y_1\cup Y_2\cup\cdots\cup Y_r$ with $r=(a-1)b+1$ so that $|H\cap Y_i|\leqslant 1$ for all edges $H\in\mathcal{H}$ and all i with $1\leqslant i\leqslant r$.*

We are now ready to present the upper bound on dimension established by Füredi and Kahn (1986).

Theorem 7.6. *Let $\mathbf{P}$ be any poset with $\Delta(\mathbf{P})\leqslant k$. Then*

$$\dim(\mathbf{P})\leqslant 50k(\log k)^2.$$

Proof. Let $n = |X|$. When $k < 500$, the result follows from the Rödl–Trotter inequality $\dim(\mathbf{P}) \leqslant 2k^2 + 2$, so we assume $k \geqslant 500$. Let $\mathscr{H}$ be the hypergraph whose vertex set is X and whose edges are the up sets $U(x) = \{y \in X \colon x < y \text{ in } \mathbf{P}\}$. By Corollary 7.4, we obtain a partition $X = X_1 \cup X_2 \cup \cdots \cup X_s$ where $s = \lceil (k+1)/\log(k+1) \rceil$ so that $|U(x) \cap X_i| \leqslant v = \lceil 4.7 \log(k+1) \rceil$ for every $x \in X$ and every $i = 1, 2, \ldots, s$.

Then let $\mathscr{H}_i$ be the hypergraph obtained by restricting $\mathscr{H}$ to X_i. Each edge in $\mathscr{H}_i$ has size at most v and no point of X_i belongs to more than $k+1$ edges. By Lemma 7.5, we obtain a partition $X_i = X_{i1} \cup X_{i2} \cup \cdots \cup X_{ir}$ where $r = (v-1)(k+1)+1$ so that $|U(x) \cap X_{ij}| \leqslant 1$ for all x, i, j. Now we know by Lemma 7.2 that

$$d = \dim(1, v+1; r) \leqslant (v+1)^2 \left(1 + \log \frac{(v-1)(k+1)+1)}{v+1}\right).$$

Let $M_1, M_2, \ldots, M_d$ be a family of linear orders on $\{1, 2, \ldots, r\}$ so that for each $(v+1)$-element subset $S \subset \{1, 2, \ldots, r\}$ and each $y \in S$, there is some i with y the least element of S in M_i. For each ij, let N_{ij} be an arbitrary linear order on X_{ij} and let $\hat{N}_{ij}$ denote the dual of N_{ij}, i.e., $u < v$ in $\hat{N}_{ij} \Longleftrightarrow v < u$ in N_{ij}. Finally, for each $i = 1, 2, \ldots, s$ and $j = 1, 2, \ldots, d$, we define two linear orders L_{ij} and L'_{ij} on X by

$$L_{ij} = N_{iM_j(1)} < N_{iM_j(2)} < \cdots < N_{iM_j(r)} < X - X_i, \text{ and}$$

$$L'_{ij} = \hat{N}_{iM_j(1)} < \hat{N}_{iM_j(2)} < \cdots < \hat{N}_{iM_j(r)} < X - X_i.$$

In both L_{ij} and L'_{ij}, the ordering on $X - X_i$ is arbitrary. Furthermore, the subscripts are interpreted so that M_j orders $\{1, 2, \ldots, r\}$ by $M_j(1) < M_j(2) < \cdots < M_j(r)$.

Now suppose $x, y \in \mathbf{P}$ with $y \not\leqslant x$ in $\mathbf{P}$. Choose α so that $x \in X_{i\alpha}$. Set $T = \{j \colon U(y) \cap X_{ij} \neq \emptyset\}$. Since $|T| \leqslant v$, there is some β for which α is the least element of $T \cup \{\alpha\}$ in M_β. If $y \notin X_{i\alpha}$, then $x < y$ in both $L_{i\beta}$ and $L'_{i\beta}$. If $y \in X_{i\alpha}$, then $x < y$ in exactly one of $L_{i\beta}$ and $L'_{i\beta}$.

This shows that $\dim(\mathbf{P}) \leqslant 2sd < 50k(\log k)^2$ as claimed. □

Until recently it was not known whether there exists an absolute constant c so that $\dim(\mathbf{P}) < ck$ whenever $\Delta(\mathbf{P}) \leqslant k$. Note that for each $k \geqslant 2$, the crown $\mathbf{S}^0_{k+1}$ satisfies $\Delta(\mathbf{S}^0_{k+1}) = k$ and $\dim(\mathbf{S}^0_{k+1}) = k+1$. However, Erdős et al. (1991) have substantially improved this lower bound by investigating the dimension of a height-two random poset. They define the sample space $\Omega(n, p)$ as consisting of all height-two posets containing n minimal points $a_1, a_2, \ldots, a_n$ and n maximal points $b_1, b_2, \ldots, b_n$. For a poset $\mathbf{P} \in \Omega(n, p)$, the probability that $a_i < b_j$ in $\mathbf{P}$ is equal to p (which in general is a function of n). Events corresponding to distinct pairs of points in $\mathbf{P}$ are independent.

Erdős et al. (1991) develop upper and lower bounds on the expected value of the dimension of a random poset for values of p in the range

$$\frac{(\log n)^{1+\varepsilon}}{n} < p < 1 - n^{-1+\varepsilon}.$$

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8. Interval orders and se

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Theorem 8.1. *A poset* $\mathbf{P}$

However, taking the particular value $p = 1/\log n$, their results imply that there exist absolute positive constants δ_1 and δ_2 so that the following inequalities almost surely hold for the random poset $\mathbf{P}$:

$$\Delta(\mathbf{P}) < \delta_1 n/\log n \quad \text{and} \quad \dim(\mathbf{P}) > \delta_2 n.$$

This shows that if we define the function $f: \mathbb{N} \to \mathbb{N}$ by "$f(k)$ is the largest positive integer for which there exists a poset $\mathbf{P} = (X, P)$ with $\Delta(\mathbf{P}) \leqslant k$ and $\dim(\mathbf{P}) \leqslant f(k)$", then there exist absolute positive constants c_1 and c_2 so that

$$c_1 k(\log k) \leqslant f(k) \leqslant c_2 k(\log k)^2.$$

It is probably a very difficult problem to determine the correct exponent on the logarithmic factor in the preceding inequality. Perhaps the answer will come with improved bounds on the expected value of the dimension of a random poset in the sparse case, i.e., for values of p satisfying $p \leqslant (\log n)^{1+\varepsilon}/n$. Other dimension-related questions for random posets are posed in Erdős et al. (1991); further applications of dimension-theoretic concepts for random posets are given in Brightwell and Trotter (1994b).

Many other interesting dimension-theoretic questions center around families of subsets ordered by inclusion. For example, the proof of Theorem 7.6 requires an estimate on the dimension of the poset formed by the 1-element and $(\log n)$-element subsets of an n-element set. Lemma 7.2 gives an upper bound of the form $c(\log n)^3$, and Kierstead (1995) has just shown a lower bound of the form $c \log^3 n/\log\log n$. So some genuinely new idea is needed to determine more accurate estimates for $f(k)$.

Dushnik (1950) developed upper and lower bounds for $\dim(1, k; n)$ which result in an exact formula when $k > 2\sqrt{n}$, and Spencer (1972) gave asymptotic results when k is relatively small in comparison to n. Füredi et al. (1991) give the following asymptotic formula:

$$\dim(1, 2 : n) = \lg\lg n + \left(\frac{1}{2} + o(1)\right) \lg\lg\lg n.$$

Hurlbert et al. (1994) show that $\dim(2, n-2; n) = n - 1$, when $n \geqslant 5$; and Brightwell et al. (1994) show that $\dim(s, s+k; n) = O(k^2 \log n)$.

8. Interval orders and semiorders

Let $\mathcal{I}$ be a collection of closed intervals of $\mathbb{R}$. Define a partial order P on $\mathcal{I}$ by $I_1 < I_2$ in $P \iff x < y$ in $\mathbb{R}$ for every $x \in I_1$, and $y \in I_2$. Posets obtained from this construction are called *interval orders*. The following theorem of Fishburn (1970) provides a forbidden subposet characterization of interval orders. We leave the proof as an exercise.

Theorem 8.1. *A poset* $\mathbf{P}$ *is an interval order* $\iff$ $\mathbf{P}$ *does not contain* $\mathbf{2} + \mathbf{2}$.

A poset **P** is called a *semiorder* if there exists a function $f: X \to \mathbb{R}$ so that $x < y$ in $\mathbf{P} \iff f(y) > f(x) + 1$. Evidently, a semiorder is an interval order having a representation in which all intervals have length 1. The following theorem due to Scott and Suppes (1958) provides a forbidden subposet characterization of semiorders. Again the proof is omitted.

Theorem 8.2. *A poset* **P** *is a semiorder* $\iff$ **P** *does not contain either* $\mathbf{2}+\mathbf{2}$ *or* $\mathbf{3}+\mathbf{1}$.

Rather than present the proofs of these two important, but by now well-known, theorems, we choose instead to discuss some recent work in which interval orders and semiorders surface in a surprising manner.

Let $\mathcal{P}$ be a class of posets. We say that the on-line dimension of $\mathcal{P}$ is at most t if there exists a strategy for constructing a realizer $L_1, L_2, \ldots, L_t$ for any poset from $\mathcal{P}$ constructed in an on-line fashion. As was discussed in section 4, the poset and the realizer are to be constructed one point at a time. At each step, a new point is added to the poset. This point is then inserted into each of the existing linear extensions in a manner such that they remain a realizer.

The reader may enjoy showing that if $\mathcal{P}$ is the set of posets of dimension at most 2, then the on-line dimension of $\mathcal{P}$ is infinite, i.e., for each t, there is no algorithm which will construct on-line realizers of size t for posets from $\mathcal{P}$.

The members of family $\{\mathbf{S}_3^k\colon k \geqslant 0\}$ of 3-dimensional crowns are called 3-*crowns*. The following result is due to Kierstead et al. (1984).

Theorem 8.3. *Let* $\mathcal{P}_n$ *denote the class of posets of width at most n which do not contain any 3-crowns. Then the on-line dimension of* $\mathcal{P}_n$ *is at most* $((5^n-1)/4)!$.

Proof. Set $t = (5^n - 1)/4$. Use Kierstead's theorem (4.2) to construct an on-line partition of a poset **P** in $\mathcal{P}$ into t chains $C_1, C_2, \ldots, C_t$. For each $x \in X$, let $C(x)$ denote the unique α for which $x \in C_\alpha$. Let M be any of the $t!$ linear orders on $\{1, 2, \ldots, t\}$. We construct on-line a linear extension L_M of a poset from $\mathcal{P}_n$. When the new point x enters **P**, let $S_1 = \{y \in \mathbf{P}\colon x < y \text{ in } \mathbf{P}\}$ and $S_2 = \{y \in \mathbf{P}\colon x \parallel y \text{ in } \mathbf{P} \text{ and } C(x) < C(y) \text{ in } M\}$. If $S_1 \cup S_2 = \emptyset$, insert x at the top of L_M. If $S_1 \cup S_2 \neq \emptyset$, and y is the lowest element of $S_1 \cup S_2$ in L_M, insert x immediately under y.

We now show that the set of all $t!$ linear extensions of **P** determined by this procedure form a realizer of **P**. To accomplish this, choose an arbitrary incomparable pair $x \parallel y$ from **P**. We show that there is at least one M for which $x > y$ in L_M. In fact, we will show that $x > y$ in some L_M so that $C(y)$ is the least element in M and $C(x)$ is the greatest element in M.

A *fence* F starting up at x and ending at y is a sequence $x = x_0, x_1, x_2, \ldots, x_i = y$ for which the only comparabilities between these points are $x_0 < x_1 > x_2 < x_3 > \cdots$. Note that such a fence starts up from $x_0 = x$, but can end either up or down at $x_i = y$.

Let $Y = \{1, 2, \ldots, t\} - \{C(x), C(y)\}$. Define a binary relation Q on Y by "$\alpha Q \beta \iff$ there exist points u, v and a fence F starting up at x and ending at v so that $\alpha = C(u), \beta = C(v), u < y$ and $\{u, y\} \parallel F$ in **P**".

Claim. *If $\alpha Q\beta$ and $\gamma Q\delta$, then either $\alpha Q\delta$ or $\gamma Q\beta$.*

Proof. Choose points u, v and a fence $F = \{x = x_0, x_1, x_2, \ldots, x_i = v\}$ which witness $\alpha Q\beta$. Similarly, choose points w, z and a fence $G = \{x = u_0, u_1, \ldots, u_j = z\}$ which witness $\gamma Q\delta$. Suppose that neither $\alpha Q\delta$ nor $\gamma Q\beta$. Then u is less than one or more points in G. Choose the least m so that $u < u_m$. Then m is odd (and positive). Similarly, let p be the least integer so that $w < x_p$. The p is odd and positive. Then the set $H = \{x, x_1, x_2, \ldots, x_p, u_1, u_2, \ldots, u_m\}$ is connected and has x_p and u_m as distinct maximal elements. Let $\mathbf{K}$ be a minimum-size connected subposet of $\mathbf{H}$ containing x_p and u_m. Then $\mathbf{K} \cup \{y, u, w\}$ is a 3-crown. The contradiction completes the proof of the claim. □

From its definition, it is clear that Q is irreflexive, i.e., we never have $\alpha Q\alpha$ for any $\alpha \in Y$. By taking $\beta = \gamma$ in the claim, we conclude that either $\alpha Q\gamma$ or $\beta Q\beta$. Hence $\alpha Q\delta$. This shows Q is transitive, and is therefore a strict partial order on Y. Furthermore, in view of Theorem 8.1, (Y, Q) is an interval order!

Choose an interval representation of (Y, Q) in which all end points are distinct. Then let M' be the linear extension of Q determined by the left end points in this representation. Form M from M' by adding $C(y)$ as least element and $C(x)$ as greatest element. We claim that $x > y$ in L_M.

Suppose to the contrary that $x < y$ in L_M. When the latter of these two points enters the poset, let $x = u_0 < u_1 < u_2 < \cdots < u_s < u_{s+1} = y$ be the sequence of points between x and y in L_σ. Note that for all $i = 0, 1, 2, \ldots, s$, either ($u_i < u_{i+1}$ in $\mathbf{P}$) or ($u_i \parallel u_{i+1}$ and $C(u_i) < C(u_{i+1})$ in M). We call any sequence $x = v_0, v_1, v_2, \ldots, v_m, v_{m+1} = y$ a *blocking chain* if

(1) $v_0 < v_1 < \cdots < v_m < v_{m+1}$ in L_M;

(2) for $i = 0, 1, \ldots, m$, either ($v_i < v_{i+1}$ in $\mathbf{P}$) or ($v_i \parallel v_{i+1}$ and $C(v_i) < C(v_{i+1})$ in M).

Of all blocking chains, we choose one, say $\{v_0, v_1, \ldots, v_{m+1}\}$, for which m is as small as possible. For each $i = 1, 2, \ldots, m$, let $\alpha_i = C(v_i)$. The minimality of m implies that $v_i \parallel v_j$ and $\alpha_i > a_j$ in M whenever $0 \leqslant i, j \leqslant m+1$ and $|j - i| \geqslant 2$. It follows that m is even and that for even i, $vi < v_{i+1}$ in X. For odd i, $v_i \parallel v_{i+1}$ and $\alpha_i < \alpha_{i+1}$ in M. Thus $C(y) < \alpha_{m-1} < \alpha_m < \alpha_{m-3} < \alpha_{m-2} < \cdots < \alpha_5 < \alpha_6 < \alpha_3 < \alpha_4 < \alpha_1 < \alpha_2 < C(x)$ in M.

Next we observe that the set $F = \{v_0, v_1\}$ is a fence starting up at x and ending at v_1. Also $\{y, v_m\} \parallel F$. It follows that $\alpha_m < \alpha_1$ in Q. Choose the largest integer i so that $\alpha_m < \alpha_i$ in Q. Suppose first that i is odd. Then $\alpha_i < \alpha_{i+1}$ in M, which implies that the left end point of the interval corresponding to α_i is less than the left end of the interval corresponding to α_{i+1}. However, the inequality $\alpha_m < \alpha_i$ in Q implies that the interval for α_m lies entirely to the left of the interval for α_i. This in turn implies $\alpha_m < \alpha_{i+1}$ in Q, contradicting our choice of i.

Now suppose that i is even. Choose points $u \in C_{\alpha_m}$, $z \in C_{\alpha_i}$ and a fence $F = \{x = z_0, z_1, z_2, \ldots, z_r = z\}$ starting up at x and ending at z so that $u < y$ and $\{u, y\} \parallel F$. Let $u' = \text{MAX}\{u, v_m\}$ and let G be a minimum-size connected subposet of $F \cup \{v_i, v_{i+1}\}$ so that G contains both x and v_{i+1}. Then G is a fence starting up at x and ending at v_{i+1}. Furthermore, $\{y, u'\} \parallel G$ which implies $\alpha_m < \alpha_{i+1}$ in Q. This is also a contradiction. □

It is mildly irritating that we do not know whether it is necessary to exclude all 3-crowns from the posets in order to have finite on-line dimension. It is certainly necessary to exclude $\mathbf{S}_3^0$, but perhaps this is enough.

We next discuss an extremal problem for interval orders with a surprising connection to hamiltonian circuit problems. It is well known that there exist posets whose cover graphs have large chromatic number [see Kříž and Nešetřil (1991), for example]. It is easy to see that such graphs exist as cover graphs of interval orders. In connection with this topic, Felsner and Trotter (1995) made the following conjecture.

Conjecture 8.4. Let $n \geqslant 1$, and let $t = 2^n$. Then there exists a permutation $A_1, \ldots, A_t$ of the subsets of $\{1, \ldots, n\}$ so that:
(1) $A_1 = \emptyset$, and
(2) For each $i = 1, 2, \ldots, t-1$, either $A_i \subset A_{i+1}$ or $A_{i+1} \subset A_i$. Furthermore, $|A_i \Delta A_{i+1}| = 1$.
(3) For each $i, j = 1, 2, \ldots, t$, if $A_i \subset A_j$ and $i > j$, then $i = j + 1$.

Here is a re-formulation of the preceding conjecture as an extremal problem.

Conjecture 8.5. For each $n \geqslant 1$, let $f(n) = s$ be the largest integer for which there exists a sequence $B_1, B_2, \ldots, B_s$ of distinct subsets of $\{1, 2, \ldots, n\}$ so that:
(1) $B_1 = \emptyset$, and
(2) for each $i = 1, 2, \ldots, s-1$, $B_{i+1} \not\subset B_i$, and
(3) for each $i = 1, 2, \ldots, s-2$, $B_{i+2} \not\subset B_i \cup B_{i+1}$.
Then $f(n) = 2^{n-1} + \lfloor (n+1)/2 \rfloor$.

Trotter and Felsner show that $f(n) \leqslant 2^{n-1} + \lfloor (n+1)/2 \rfloor$ and that equality holds if and only if Conjecture 8.4 is valid. These conjectures are related to the following (surprisingly difficult) problem.

Conjecture 8.6. Let $\mathbf{G}$ denote the comparabilty graph of the poset formed by the k-element and $(k+1)$-element subsets of a $(2k+1)$-element set partially ordered by inclusion. Then $\mathbf{G}$ has a hamiltonian cycle.

Although Conjecture 8.6 is known to be true for small values of k, most of the results thus far are negative, i.e., a hamiltonian cycle *cannot* be formed by combining certain types of matchings [see Duffus et al. (1988) and Kierstead and Trotter (1988), for example]. On the other hand, it is shown in Felsner and Trotter (1995) that there exists a cycle whose size is at least one-fourth of the total number of vertices. This fraction has subsequently been raised by C. Savage and P. Winkler (pers. comm).

We next discuss an elementary extremal problem for posets. For integers n, k with $0 \leqslant k \leqslant \binom{n}{2}$, let $Q(n, k)$ denote the class of all posets having n points and

k comparable pairs. For
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Trotter (1992) then show

Theorem 8.7. *Every pos*

Proof. Let **P** be such a p
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$z < y$ by $z < x$ for all z
$e(\mathbf{P}') > e(\mathbf{P})$.
Exchanging x and y m
1–1, it is not onto since a
image of the map. The co
If **P** contains a 3-elem
ble to w, form $\mathbf{P}''$ from
$U(y) - U(w)$. As before,

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Now let n and k be fixed
that among all posets con
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Problem 8.8. For fixed n
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The next result is a re
conjecture for semiorders

k comparable pairs. For each poset $\mathbf{P} \in Q(n,k)$, let $e(\mathbf{P})$ count the number of linear extensions of $\mathbf{P}$. Then set $e(n,k) = \max\{e(\mathbf{P})\colon \mathbf{P} \in Q(n,k)\}$. Fishburn and Trotter (1992) then show that the extremal posets are semiorders.

Theorem 8.7. *Every poset* $\mathbf{P} \in Q(n,k)$ *with* $e(\mathbf{P}) = e(n,k)$ *is a semiorder.*

Proof. Let $\mathbf{P}$ be such a poset. We first show that $\mathbf{P}$ does not contain $\mathbf{2}+\mathbf{2}$. Suppose to the contrary that the chains $u < x$ and $v < y$ are incomparable. Of all such pairs of chains, we choose two for which $|U(x)| + |U(y)|$ is minimum. We may therefore assume that $U(x) \subset U(y)$. Let $\mathbf{P}'$ be the poset obtained by replacing the relations $z < y$ by $z < x$ for all $z \in D(y) - D(x)$. Then $\mathbf{P}' \in Q(n,k)$. We now show that $e(\mathbf{P}') > e(\mathbf{P})$.

Exchanging x and y maps $\mathscr{E}(\mathbf{P}) - \mathscr{E}(\mathbf{P}')$ to $\mathscr{E}(\mathbf{P}') - \mathscr{E}(\mathbf{P})$. Although the map is 1–1, it is not onto since any linear extension in which $y < u < v < x$ is not in the image of the map. The contradiction shows $\mathbf{P}$ is an interval order.

If $\mathbf{P}$ contains a 3-element chain $x < y < z$ with all three points incomparable to w, form $\mathbf{P}''$ from $\mathbf{P}$ by replacing the relations $y < z$ by $w < z$ for all $z \in U(y) - U(w)$. As before, $e(\mathbf{P}'') > e(\mathbf{P})$. The contradiction shows $\mathbf{P}$ is a semiorder. □

At first glance, Theorem 8.7 seems to be very helpful in determining $e(n,k)$, and some progress is made in Fishburn and Trotter (1992). However, the general problem remains open.

Recently, P. Winkler (pers. comm.) has proposed another extremal problem involving semiorders. For a poset $\mathbf{P} = (X,P)$ and a point $x \in X$, let $\deg(x) = |\{y \in X\colon y < x \text{ in } P, \text{ or } y > x \text{ in } P\}|$. Then define the *flexibility* of $\mathbf{P}$, denoted $\mathrm{flex}(\mathbf{P})$, by

$$\mathrm{flex}(\mathbf{P}) = \sum_{x\in X} (\deg(x))^2.$$

Now let n and k be fixed integers with $0 \leqslant k \leqslant \binom{n}{2}$. It is an easy exercise to show that among all posets containing n points and k comparable pairs, any poset with maximum flexibility is a semiorder.

Problem 8.8. For fixed n and k, find all semiorders with n points and k comparable pairs for which the flexibility is maximum.

Problem 8.9. For a poset $\mathbf{P} = (X,P)$ with n points and k comparable pairs, let i be an integer with $0 \leqslant i \leqslant k$ and let a_i denote the number of permutations (linear orders, not necessarily linear extensions) of the ground set X so that exactly i of the k comparable pairs are in the same order as in P. Then $\sum_{i=0}^{k} a_i = n!$, and $a_i = a_{k-i}$, for each $i = 0,1,\ldots,k$. Is the sequence $a_0, a_1, \ldots, a_k$ unimodal?

Winkler noted that the sequence need not be log-concave, so the mixed volumes approach used by Stanley (see the discussion in section 6) will not apply.

The next result is a recent theorem of Brightwell (1989) establishing the $\frac{1}{3}$–$\frac{2}{3}$ conjecture for semiorders.

Theorem 8.10. *Let* $\mathbf{P} = (X, P)$ *be a semiorder which is not a chain. Then* X *contains a pair* x, y *of incomparable points with* $\frac{1}{3} \leqslant \text{Prob}[x < y] \leqslant \frac{2}{3}$.

Proof. Suppose the result is false and choose a counterexample with $|X|$ minimum. Define a linear extension L by $x < y \iff \text{Prob}[x < y] > \frac{2}{3}$. Let $|X| = n$ and label the points in X so that $x_1 < x_2 < \cdots < x_n$ in L.

Since $\mathbf{P}$ is a semiorder, there exists a function $f: X \to \mathbb{R}$ so that $x < y$ in $\mathbf{P} \iff f(y) > f(x) + 1$. We now show that $f(x_1) < f(x_2) < \cdots < f(x_n)$. Suppose to the contrary that $1 \leqslant i < j \leqslant n$, but that $f(x_j) < f(x_i)$. Then $x_i \not< x_j$ in $\mathbf{P}$. However, $x_i < x_j$ in L, so $x_j \not< x_i$ in $\mathbf{P}$. Thus $x_i \parallel x_j$ in $\mathbf{P}$. However, the inequality $f(x_j) < f(x_i)$ implies that $u > x_j$ in $\mathbf{P}$ whenever $u > x_i$ in $\mathbf{P}$. Dually $v < x_i$ in $\mathbf{P}$ whenever $v < x_j$ in $\mathbf{P}$.

It follows that every $L \in \mathscr{E}(\mathbf{P})$ with $x_i < x_j$ in L can be transformed into a linear extension L' with $x_j < x_i$ in L' just by interchanging these two points. The mapping is 1–1 which shows $\text{Prob}[x_j < x_i] \geqslant \frac{1}{2}$. Thus $\text{Prob}[x_j < x_i] > \frac{2}{3}$ and $x_j < x_i$ in L. The contradiction shows $f(x_1) < f(x_2) < \cdots < f(x_n)$ as claimed.

We now show that $x_i \parallel x_{i+1}$ for all $i = 1, 2, \ldots, n-1$. Suppose to the contrary that $x_i < x_{i+1}$ in $\mathbf{P}$. Then every point of $X' = \{x_1, x_2, \ldots, x_i\}$ is less than every point of $X'' = \{x_{i+1}, x_{i+1}, \ldots, x_n\}$. At least one of $\mathbf{P}'$ and $\mathbf{P}''$ is not a chain, and we can restrict our attention to that subposet to locate x and y. The contradiction shows $x_i \parallel x_{i+1}$ for $i = 1, 2, \ldots, n-1$ as claimed.

We say x_j *separates* x_i *and* x_{i+1} *from above* if $x_j :> x_i$ and $x_j \parallel x_{i+1}$. Dually we say x_j *separates* x_i *and* x_{i+1} *from below* if $x_j <: x_{i+1}$ and $x_j \parallel x_i$. We say x_j *separates* x_i *and* x_{i+1} if it either separates them from above or it separates them from below.

If x_j separates x_i and x_{i+1} from above, then $x_k < x_j$ in $\mathbf{P}$ for $k = 1, 2, \ldots, i$. This implies that x_j does not separate x_k and x_{k+1} when $1 \leqslant k < i$. Dually, if x_j separates x_i and x_{i+1} from below, then x_j does not separate x_k and x_{k+1} when $i < k < n$. So each x_j separates at most one pair from below and at most one pair from above.

However, x_1 and x_2 cannot separate pairs from above and x_{n-1} and x_n cannot separate any pair from below. If follows that there are at most $2(n-4) + 4 = 2n - 4$ ordered pairs (i, j) so that x_j separates x_i and x_{i+1}. Hence there is at least one (in fact at least two) values of i with $1 \leqslant i < n$ for which there is at most one j so that x_j separates x_i and x_{i+1}.

For such a value of i, partition $\mathscr{E}(\mathbf{P})$ into three sets by letting $\mathscr{E}_1 = \{L \in \mathscr{E}(\mathbf{P})$: $x_i < x_{i+1}$ in L, and no element separating x_i and x_{i+1} is between them in $L\}$; $\mathscr{E}_2 = \{L \in \mathscr{E}(\mathbf{P})$: $x_i < x_{i+1}$ and $L \notin \mathscr{E}_1\}$, and $\mathscr{E}_3 = \{L \in \mathscr{E}(\mathbf{P})$: $x_{i+1} < x_i$ in $L\}$. Now let $t = |\mathscr{E}(\mathbf{P})|$. Then $|\mathscr{E}_1| + |\mathscr{E}_2| > 2t/3$.

If $|\mathscr{E}_2| \geqslant t/3$, let j be the unique integer so that x_j separates x_i and x_{i+1}. Then x_j is between x_i and x_{i+1} in every $L \in \mathscr{E}_3$. If $j > i+1$, this implies $\text{Prob}[x_j < x_{i+1}] \geqslant \frac{1}{3}$, and if $j < i$, it implies $\text{Prob}[x_i < x_j] \geqslant \frac{1}{3}$. Both of these implications are false, so we know $|\mathscr{E}_2| < t/3$. Thus $|\mathscr{E}_1| > t/3$.

Now let $L \in \mathscr{E}_1$. Form L' from L by interchanging x_i and x_{i+1}. This interchange is possible since any point between x_i and x_{i+1} in L is incomparable with both. This procedure determines a 1–1 map from $\mathscr{E}_1$ to $\mathscr{E}_3$. However, no such map exists because $|\mathscr{E}_3| < t/3$. The contradiction completes the proof. □

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9. Degrees of freedom

Given a family $\mathscr{F}$ of sets,
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Fishburn and Trotter (
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The connection between semiorders and the $\frac{1}{3}$–$\frac{2}{3}$ conjecture is even more complex than suggested by the preceding result. Consider the infinite poset $\mathbf{P} = (X, P)$, where $X = \{x_i\colon i \in \mathbb{Z}\}$. Furthermore, we define $x_i < x_j$ in P if and only if $i < j - 1$ in $\mathbb{Z}$. Then $\mathbf{P}$ is a width-two semiorder. Also observe that any incomparable pair in $\mathbf{P}$ is of the form (x_i, x_{i+1}), for some $i \in \mathbb{Z}$. For a positive integer n, let $\mathbf{P}_n$ denote the subposet of $\mathbf{P}$ determined by the points whose subscripts in absolute value are at most n. Then it is an easy exercise to show that

$$\lim_{n\to\infty} \operatorname{Prob}[x_0 > x_1] = (5 - \sqrt{5})/10.$$

Note that $(5 - \sqrt{5})/10 \approx 0.2764 < \frac{1}{3}$. So the $\frac{1}{3}$–$\frac{2}{3}$ conjecture is *false* for infinite width-two semiorders, even though it is true for any finite poset which is either a width-two poset or is a semiorder! Also note that this example shows that the inequality in Theorem 6.7 is best possible when considering infinite posets of bounded width. Further results on infinite posets and balanced pairs are given in the recent papers Brightwell (1988, 1993).

Rabinovitch (1978) showed that the dimension of a semiorder is at most 3, but that an interval order may have arbitrarily large dimension. Füredi et al. (1991) showed that if $\mathbf{P} = (X, P)$ is an interval order of height n, then

$$\dim(\mathbf{P}) \leqslant \lg\lg n + \left(\tfrac{1}{2} + o(1)\right) \lg\lg\lg n.$$

This inequality is best possible. For an integer $n \geqslant 2$, let $\mathbf{I}_n$ denote the *canonical interval order* consisting of all intervals with integer end points from $\{1, 2, \ldots, n\}$. Then Füredi et al. (1991) showed that

$$\dim(\mathbf{I}_n) = \lg\lg n + \left(\tfrac{1}{2} + o(1)\right) \lg\lg\lg n.$$

This formula is closely related to the asymptotic formula given in section 7 for the dimension of the poset formed by the 1-element and 2-element subsets of $\{1, 2, \ldots, n\}$.

9. Degrees of freedom

Given a family $\mathscr{F}$ of sets, a poset $\mathbf{P}$ is called an *$\mathscr{F}$-inclusion order* if there exists a mapping which assigns to each $x \in X$ a set $S(x) \in \mathscr{F}$ so that $x \leqslant y$ in $\mathbf{P} \iff S(x) \subset S(y)$. As an example, if $\mathscr{F}$ is the collection of all closed intervals of $\mathbb{R}$, then the $\mathscr{F}$-inclusion orders are exactly the posets with dimension at most 2.

Fishburn and Trotter (1990) studied the class of *angle orders*. These are the posets which arise when $\mathscr{F}$ is the set of angular regions in the Euclidean plane, i.e., convex regions bounded by two rays emanating from a common point. They proved that every interval order is an angle order, as is every poset of dimension at most four. Both results admit elementary proofs, but while the first result may be mildly surprising, the second is certainly not. In a certain sense, to specify an angle requires four coordinates – two to locate the corner point and one for each ray to specify the angle from $[0, 2\pi)$ at which it leaves the corner point.

Fishburn and Trotter conjectured that not all 5-dimensional posets are angle orders, but were only able to prove the existence of a 7-dimensional poset which is not an angle order. R. Jamison (pers. comm.) settled this conjecture in the affirmative with an intricate ad hoc argument. However, Alon and Scheinerman (1988) have produced a much more general result using a powerful theorem of Warren (1968). We now outline their approach.

For $x \in \mathbb{R}$, let $\operatorname{sgn}(x) = +$ if $x \geqslant 0$ and $\operatorname{sgn}(x) = -$ if $x < 0$. For a vector $\mathbf{x} = (x_1, x_2, \ldots, x_t)$, $\operatorname{sgn}(\mathbf{x})$ denotes the vector $(\operatorname{sgn}(x_1), \operatorname{sgn}(x_2), \ldots, \operatorname{sgn}(x_t))$. The vector $\operatorname{sgn}(\mathbf{x})$ is called the *sign pattern* of $\mathbf{x}$ and t is the *length* of the pattern. We say $\mathcal{F}$ has *at most k degrees of freedom*, and write $\deg(\mathcal{F}) \leqslant k$, if the following conditions are satisfied:

(1) there exists a mapping F which assigns to each $S \in \mathcal{F}$ a k-tuple $F(S) = (S(1), S(2), \ldots, S(k))$ from $\mathbb{R}^k$;

(2) there exists a finite set $P_1, P_2, \ldots, P_t$ of polynomials in $2k$ variables $x_1, x_2, \ldots, x_{2k}$; and

(3) there exists a set J of sign patterns of length t so that for every pair S, T of sets from $\mathcal{F}$, $S \subset T \iff \operatorname{sgn}(\mathbf{y}(S, T)) \in J$ where $\mathbf{y}(S, T)$ is the vector of length t whose jth coordinate is given by $P_j(S(1), S(2), \ldots, S(k), T(1), T(2), \ldots, T(k))$.

To illustrate this definition, let $\mathcal{F}$ denote the set of closed disks in $\mathbb{R}^2$. With each set (disk) S, we take $(S(1), S(2))$ as the coordinates of the center of S and $S(3)$ as the radius. Take $P_1 = x_6 - x_3$ and $P_2 = (x_6 - x_3)^2 - (x_5 - x_2)^2 - (x_4 - x_1)^2$. Then take $J = \{(+, +)\}$. Let S and T be disks. Then $S \subset T \iff$ the distance from the center of S to the center of T plus the radius of S is less than or equal to $T \iff \operatorname{sgn}(\mathbf{y}(S, T)) \in J$. This shows $\deg(\mathcal{F}) \leqslant 3$, i.e., $\mathcal{F}$ has at most three degrees of freedom. As a second example, the set $\mathcal{A}$ of angular regions in $\mathbb{R}^2$ has at most four degrees of freedom. Here is Warren's theorem (Warren 1968).

Theorem 9.1. *Let $P_1, P_2, \ldots, P_t$ be polynomials in m variables and let d denote the maximum degree among these polynomials. Then there are at most $(4edt/m)^m$ sign patterns of the form $\operatorname{sgn}(P_1(\mathbf{x}), P_2(\mathbf{x}), \ldots, P_t(\mathbf{x}))$ where $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ ranges over $\mathbb{R}^m$.*

Let $P(n, k)$ denote the number of labelled posets on n points having dimension at most k. Clearly $P(n, k) \leqslant (n!)^k \leqslant n^{kn}$. Subsequent arguments will require the following lower bound on $P(n, k)$ due to Alon and Scheinerman (1988). We leave the proof as an exercise.

Theorem 9.2. *The number $P(n, k)$ of labelled posets on n points having dimension at most k satisfies:*

$$P(n, k) \geqslant (n/\log n)^{nk - 2k^2 n/\log n}.$$

The preceding two results combine easily to prove the following striking result of Alon and Scheinerman (1988).

Theorem 9.3. *Let $\mathcal{F}$ be*
Then there exists a poset

Proof. Suppose $\deg(\mathcal{F})$
and the set J of test patt
nomials. We show that wh
points having dimension a
When $\mathbf{P}$ is a labelled
an $\mathcal{F}$-order, we designat
$(S_1, S_2, \ldots, S_n)$. Each S_i is
ordered pair (i, j) with
$S_j \iff \operatorname{sgn}(\mathbf{y}(S_i, S_j)) \in J$.
$P_\alpha(x_{i1}, x_{i2}, \ldots, x_{ik}, x_{j1}, x_{j2},$
Concatenate in lexicogr
vector $\mathbf{y}$ of length $n(n -$
entries are determined by
where $1 \leqslant i \leqslant n$ and $1 \leqslant$
If $(T_1, T_2, \ldots, T_n)$ is an
sign pattern $\operatorname{sgn}(\mathbf{y})$ as $(S_1$
same labelled poset $\mathbf{P}$. By
$(4edn(n-1)t/nk)^{nk}$ possi
than $P(n, k+1)$ when n is

There are a number of p
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Given a point x in d-dim
$B_d(\mathbf{x}, r)$ denote the ball of
at most r from $\mathbf{x}$. It is cus
$\mathbf{P} = (X, P)$ is called a *d-di*
d-dimensional sphere B_x s

Problem 9.4. If $\mathbf{P}$ is a fin
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every interval order is a cir
poset is a circle order; in f
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Problem 9.5. If $\mathbf{P}$ is a fin

Problem 9.5 is intriguing
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hand, it is a relatively easy
n-gons in the plane (with
$\dim(\mathbf{P}) \leqslant 3$, then $\mathbf{P}$ is a $\mathcal{F}_n$

Theorem 9.3. *Let $\mathcal{F}$ be any family of sets having at most k degrees of freedom. Then there exists a poset* $\mathbf{P}$ *with* $\dim(\mathbf{P}) \leqslant k+1$ *which is not an $\mathcal{F}$-order.*

Proof. Suppose $\deg(\mathcal{F}) \leqslant k$ is witnessed by the set $P_1, P_2, \ldots, P_t$ of polynomials and the set J of test patterns. Let d denote the maximum degree of these polynomials. We show that when n is sufficiently large, there is a labelled poset on n points having dimension at most $k+1$ which is not an $\mathcal{F}$-order.

When $\mathbf{P}$ is a labelled poset on the ground set $X = \{1, 2, \ldots, n\}$ and $\mathbf{P}$ is an $\mathcal{F}$-order, we designate the sets in $\mathcal{F}$ corresponding to the points in X by $(S_1, S_2, \ldots, S_n)$. Each S_i is associated with a vector $\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots, x_{ik})$. For each ordered pair (i,j) with $1 \leqslant i,\ j \leqslant n$ and $i \neq j$, we know $i < j$ in $\mathbf{P} \iff S_i \subset S_j \iff \operatorname{sgn}(\mathbf{y}(S_i, S_j)) \in J$. Recall that the αth coordinate of the vector $\mathbf{y}(S_i, S_j)$ is $P_\alpha(x_{i1}, x_{i2}, \ldots, x_{ik}, x_{j1}, x_{j2}, \ldots, x_{jk})$.

Concatenate in lexicographic order the $n(n-1)$ vectors $\mathbf{y}(S_i, S_j)$ into a single vector $\mathbf{y}$ of length $n(n-1)t$. Then $\operatorname{sgn}(\mathbf{y})$ is the sign pattern of a vector whose entries are determined by a family of $n(n-1)t$ polynomials in nk variables $x_{i\beta}$ where $1 \leqslant i \leqslant n$ and $1 \leqslant \beta \leqslant k$.

If $(T_1, T_2, \ldots, T_n)$ is another n-tuple of sets from $\mathcal{F}$ which yields the same sign pattern $\operatorname{sgn}(\mathbf{y})$ as $(S_1, S_2, \ldots, S_n)$, then these two n-tuples correspond to the same labelled poset $\mathbf{P}$. By Warren's theorem, we conclude that there are at most $(4edn(n-1)t/nk)^{nk}$ possible sign patterns. However, this number is clearly less than $P(n, k+1)$ when n is sufficiently large. □

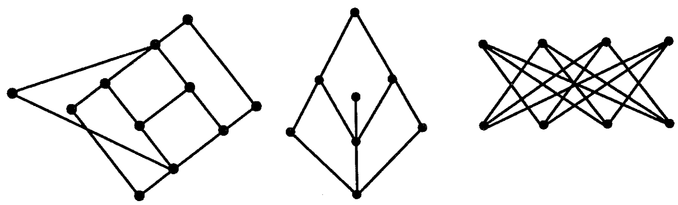
There are a number of perplexing open problems involving the representation of posets as a family of sets ordered by inclusion. Here are two of the most appealing. Given a point x in d-dimensional Euclidean space and a positive number r, let $B_d(\mathbf{x}, r)$ denote the ball of radius r centered at $\mathbf{x}$, i.e., the set of points at distance at most r from $\mathbf{x}$. It is customary to call $B_d(\mathbf{x}, r)$ a *d-dimensional sphere.* A poset $\mathbf{P} = (X, P)$ is called a *d-dimensional sphere order* if for each $x \in X$, there exists a d-dimensional sphere B_x so that $x \leqslant y$ in P if and only if $B_x \subseteq B_y$, for all $x, y \in X$.

Problem 9.4. If $\mathbf{P}$ is a finite poset, does there always exist a positive integer d so that $\mathbf{P}$ is a d-dimensional sphere order?

For historical reasons, a 2-dimensional sphere order is called a *circle order*, although it might be better to call it a *disk order*. Fishburn (1988) proved that every interval order is a circle order. Also, it is easy to see that every 2-dimensional poset is a circle order; in fact, we may require that the centers of the circles used in the representation be collinear. On the other hand, by Theorem 9.3, there exists a 4-dimensional poset which is not a circle order.

Problem 9.5. If $\mathbf{P}$ is a finite poset and $\dim(\mathbf{P}) \leqslant 3$, is $\mathbf{P}$ a circle order?

Problem 9.5 is intriguing because Scheinerman and Weirman (1989) showed that the countably infinite 3-dimensional poset $\mathbb{Z}^3$ is not a circle order. On the other hand, it is a relatively easy exercise to show that if $\mathcal{F}_n$ is the family of all regular n-gons in the plane (with bottom side horizontal) and $\mathbf{P}$ is any finite poset with $\dim(\mathbf{P}) \leqslant 3$, then $\mathbf{P}$ is a $\mathcal{F}_n$-inclusion order, for all $n \geqslant 3$.



(a) (b) (c)

Figure 10.1.

10. Dimension and planarity

A poset **P** is said to be *planar* if it has a planar Hasse diagram. The poset shown in fig. 10.1a is nonplanar, but the posets in figs. 10.1b and 10.1c are planar. Note that the diagram for the last example can be redrawn without edge crossings.

As is well known, a planar poset **P** having a greatest and least element has dimension at most 2. Kelly and Rival (1975) provide a forbidden subposet characterization of nonplanar lattices by providing a minimum list $\mathscr{L}$ of lattices so that a lattice **P** is planar if and only if **P** contains a lattice from $\mathscr{L}$ as a subposet. One lattice from $\mathscr{L}$ is shown in fig. 10.1a. The lengthy argument for this theorem must be cleverly organized just so it can be written down on a finite number of pages. There are several other theorems in dimension theory for posets which exhibit these same characteristics: Kelly's (1977) determination of all 3-irreducible posets, Trotter's (1981) determination of all 3-interval irreducible posets of height 2, and Kimble's (1973) proof that if $n \geqslant 4$ and $|X| \leqslant 2n+1$, then $\dim(\mathbf{P}) < n$ unless **P** contains the standard n-dimensional poset $\mathbf{S}_n^0$. In fact, Gallai's (1967) forbidden subgraph characterization of comparability graphs belongs in this same grouping – especially in view of its value in obtaining a list of all 3-irreducible posets (see Trotter 1992, Trotter and Moore 1976).

Planar posets can have dimension exceeding 2: the planar posets in figs. 10.1b and 10.1c have dimensions 3 and 4 respectively. Trotter and Moore (1977) proved that a planar poset having either a greatest or least element has dimension at most 3. Kelly (1981) then constructed planar posets of arbitrary dimension by the device of embedding $\mathbf{S}_n^0$ in a planar poset. Kelly's construction is illustrated in fig. 10.2.

Three interesting problems remain. Do there exist irreducible planar posets of arbitrarily large dimension? Provide a characterization of planar posets in terms of forbidden subdiagrams. Develop a fast algorithm which will produce a planar drawing of the Hasse diagram of a poset if such a drawing exists.

Recently, Schnyder (1989) produced a striking theorem relating dimension and planarity in a different manner. Let $\mathbf{G} = (V, E)$ be an ordinary undirected

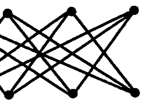
graph. We associate with
MAX(**P**) $= E$. Also verte
We call **P** the *incidence p*

Theorem 10.1. *Let* **G** *b*
incidence poset is at most

Proof. Let $\mathbf{P} = \mathbf{P}(\mathbf{G})$ be
We show that **G** is plan
argument we give for thi
(1981).]

Choose an embedding c
$\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$ so th
$y \in X \cup E$, let $\pi(y)$ be the
in $\mathbb{R}^3$. Without loss of ge
the plane $x_1 + x_2 + x_3 = 0$,

For each $u \in X$ and e
$\pi(e)$ with a straight line s
$u, v \in V$ and distinct edg
is an end point of f but r
segment $\pi(v)\pi(f)$ at a p
segment **ue** in $\mathbb{R}^3$ so that
in $\mathbb{R}^3$ so that $\pi(\mathbf{w}) = \mathbf{p}$. T
implies $\mathbf{u} \leqslant \mathbf{z} \leqslant \mathbf{w} \leqslant \mathbf{f}$, wh
$\mathbf{w} \leqslant \mathbf{z}$ implies $\mathbf{v} \leqslant \mathbf{e}$ which



(c)

diagram. The poset shown
and 10.1c are planar. Note
vithout edge crossings.
est and least element has
forbidden subposet charac-
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nent for this theorem must
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y for posets which exhibit
n of all 3-irreducible posets,
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ll 3-irreducible posets (see

planar posets in figs. 10.1b
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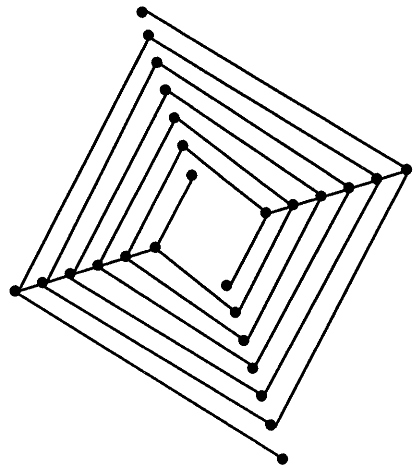

Figure 10.2.

graph. We associate with **G** a poset $\mathbf{P} = \mathbf{P}(\mathbf{G})$ of height 2. In **P**, $\mathrm{MIN}(\mathbf{P}) = V$ and $\mathrm{MAX}(\mathbf{P}) = E$. Also vertex x is less than edge e in $\mathbf{P} \Longleftrightarrow x$ is an end point of e. We call **P** the *incidence poset* of **G**. Here is Schnyder's theorem.

Theorem 10.1. *Let* **G** *be a graph. Then G is planar $\Longleftrightarrow$ the dimension of its incidence poset is at most* 3.

Proof. Let $\mathbf{P} = \mathbf{P}(\mathbf{G})$ be the incidence poset of **G**. Suppose first that $\dim(\mathbf{P}) \leqslant 3$. We show that **G** is planar. Suppose to the contrary that **G** is nonplanar. [The argument we give for this part is patterned after a proof of Babai and Duffus (1981).]

Choose an embedding of **P** in $\mathbb{R}^3$ which associates with each $y \in V \cup E$ a vector $\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$ so that $u \leqslant v$ in $\mathbf{P} \Longleftrightarrow u_i \leqslant v_i$ in $\mathbb{R}$ for $i = 1, 2, 3$. For each $y \in X \cup E$, let $\pi(y)$ be the orthogonal projection of **y** on the plane $x_1 + x_2 + x_3 = 0$ in $\mathbb{R}^3$. Without loss of generality, all points in $X \cup E$ project to distinct points on the plane $x_1 + x_2 + x_3 = 0$, and these points are in general position.

For each $u \in X$ and each $e \in E$ containing u as an end point, join $\pi(u)$ and $\pi(e)$ with a straight line segment. Since **G** is nonplanar, there exist distinct vertices $u, v \in V$ and distinct edges $e, f \in E$ so that u is an end point of e but not of f, v is an end point of f but not of e, and the line segment $\pi(u)\pi(e)$ crosses the line segment $\pi(v)\pi(f)$ at a point **p** interior to both. Let **z** be the point on the line segment $\mathbf{u}e$ in $\mathbb{R}^3$ so that $\pi(\mathbf{z}) = \mathbf{p}$. Also let **w** be the point on the line segment $\mathbf{v}f$ in $\mathbb{R}^3$ so that $\pi(\mathbf{w}) = \mathbf{p}$. Then either $\mathbf{z} \leqslant \mathbf{w}$ in $\mathbb{R}^3$ or $\mathbf{w} \leqslant \mathbf{z}$ in $\mathbb{R}^3$. However, $\mathbf{z} \leqslant \mathbf{w}$ implies $\mathbf{u} \leqslant \mathbf{z} \leqslant \mathbf{w} \leqslant \mathbf{f}$, which is false since u is not an end point of f. Similarly $\mathbf{w} \leqslant \mathbf{z}$ implies $\mathbf{v} \leqslant \mathbf{e}$ which is also false. The contradiction shows that **G** is planar.

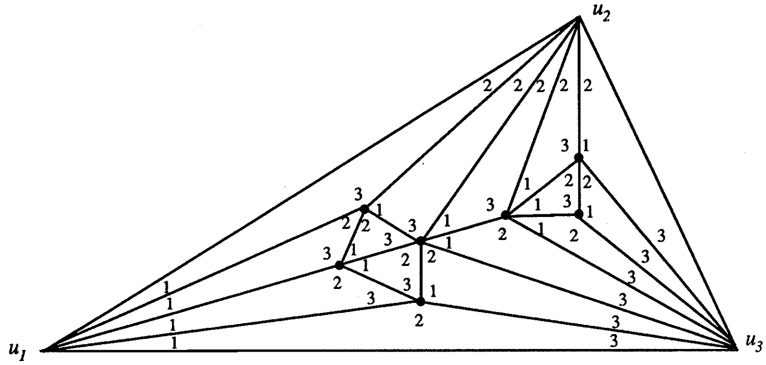

Figure 10.3.

Now suppose that **G** is planar. We show that **P** has dimension at most 3. Without loss of generality, we assume that **G** is maximal planar. Choose a planar diagram of **G** using straight line segments for the edges. This diagram is a triangulation T of the plane. Each interior region is a triangle, and T has three exterior vertices which we label in clockwise order u_1, u_2, and u_3.

Now consider a function f which assigns to each angle of each interior triangle of T a color selected from $\{1,2,3\}$. The function f is called a *normal* coloring of $T \iff$

(1) all angles incident with exterior vertex u_i are mapped by f to color i for $i = 1,2,3$;

(2) at each interior vertex u of T, there is an angle mapped by f to color i for $i = 1,2,3$;

(3) at each interior vertex u of T, all angles mapped by f to color i are consecutive for $i = 1,2,3$;

(4) at each interior vertex u of T the block of angles mapped by f to color 2 appears immediately after the block of angles mapped by f to color 1; and

(5) for each elementary triangle of T, f assigns the three angles to colors 1, 2, and 3 in clockwise order.

We illustrate this definition in fig. 10.3 with a normal coloring of a triangulation.

The following claim yields to a straightforward inductive argument and its proof is left as an exercise.

Claim 1. *Every planar triangulation has a normal coloring.*

Let C be a cycle in a planar triangulation T which has been colored normally. A vertex x belonging to C is called a *Type i* vertex on C if all angles incident with x and interior to C are colored i. When C is exterior triangle, u_i is a Type i vertex on C.

Claim 2. *If C is a cycle*

Proof. Suppose the clai
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elementary triangle. Now
Suppose that C has two
an edge $e = xy$ interior t
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and C'' both have a Type
x is a Type 1 vertex for C
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Now let $C = \{x_1, x_2, \ldots$
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C. Let C_i be the cycle ob
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It follows that one of x
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1 on C. Thus the angle of
x_{i+1} is not Type 1 for C_i.
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and x_i is not Type 1 for
If some vertex x_{i+1} is
either x_i is Type 1 for C_i f
In the first case, there is
3 vertex on C_2. The cont

Claim 3. *Let P_i be the*
$xP_iy \iff$ there exists an
the angle incident at y is
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Proof. It suffices to sho
since a directed cycle in
vertex. □

For each $i = 1,2,3$, let
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(1) The restriction of
(2) For each $e \in E$, the
than e in **P**.
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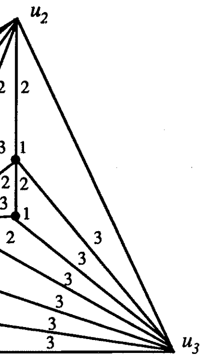
as possible. To complete



nension at most 3. Without
Choose a planar diagram
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angle, u_i is a Type i vertex

Claim 2. *If C is a cycle in T, then C contains a Type i vertex, for each $i = 1, 2, 3$.*

Proof. Suppose the claim is false. Choose a counterexample C containing the minimum number of elementary triangles. Clearly C is not the boundary of an elementary triangle. Now suppose C does not have a Type 1 vertex.

Suppose that C has two nonconsecutive vertices x and y which are adjacent via an edge $e = xy$ interior to C. Then the region bounded by C can be partitioned into regions bounded by cycles C' and C'' having e as a common edge. Now C' and C'' both have a Type 1 vertex. If x is a Type 1 vertex for C' and for C'', then x is a Type 1 vertex for C. An analogous statement holds for y. We conclude that one of x and y is a Type 1 vertex for C' and the other is a Type 1 vertex for C''. Consideration of the two elementary triangles sharing the edge shows this is impossible.

Now let $C = \{x_1, x_2, \ldots, x_s\}$ and let x_i and x_{i+1} be any two consecutive vertices of C and let z_i be the vertex so that $x_i x_{i+1} z_i$ is an elementary triangle interior to C. Let C_i be the cycle obtained by deleting the edge $x_i x_{i+1}$ and adding the edges $x_i z_i$ and $z_i x_{i+1}$. Then C_i has a Type 1 vertex because it contains fewer elementary triangles than C. Clearly z_i cannot be a Type 1 vertex on C_i because z_i is an interior vertex of T.

It follows that one of x_i and x_{i+1} is a Type 1 vertex on C_i. If x_i is Type 1 on C_i, then the angle of triangle $x_i x_{i+1} z_i$ incident with x_i must be colored 3; else x_i is Type 1 on C. Thus the angle of $x_i x_{i+1} z_i$ incident with x_{i+1} is colored 1. This implies that x_{i+1} is not Type 1 for C_i. Dually, if x_{i+1} is Type 1 for C_i, then the angle of $x_i x_{i+1} z_i$ incident with x_{i+1} is colored 2, the angle of $x_i x_{i+1} z_i$ incident with x_i is colored 1, and x_i is not Type 1 for C_i.

If some vertex x_{i+1} is Type 1 for both C_i and C_{i+1}, then x_i is Type 1 for C. So either x_i is Type 1 for C_i for $i = 1, 2, \ldots, s$, or x_{i+1} is Type 1 for C_i for $i = 1, 2, \ldots, s$. In the first case, there is no Type 2 vertex on C_1; in the second, there is no Type 3 vertex on C_2. The contradiction completes the proof. □

Claim 3. *Let P_i be the binary relation on the set V of vertices of* **G** *defined by $xP_iy \iff$ there exists an elementary triangle T having x and y as vertices in which the angle incident at y is colored i. Then the transitive closure $Q_i = \bar{P}_i$ is a partial order on X.*

Proof. It suffices to show Q_i has no directed cycles. This follows from Claim 2 since a directed cycle in Q_i could not have either a Type $i+1$ or a Type $i+2$ vertex. □

For each $i = 1, 2, 3$, let M_i be a linear extension of Q_i. Then let L_i be any linear extension of **P** so that:

(1) The restriction of L_i to V is M_i.

(2) For each $e \in E$, the M_i-largest element of V which is less than e in M_i is less than e in **P**.

Alternatively, L_i is obtained from M_i by inserting the elements of E as low as possible. To complete the proof, it suffices to show that $P = L_1 \cap L_2 \cap L_3$. To

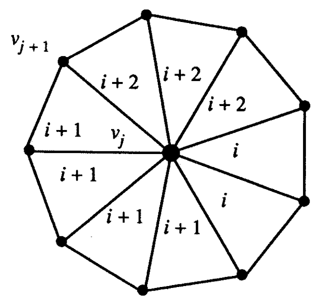

Figure 10.4.

accomplish this, it is enough to show that for each edge $e = xy$ and each vertex z not an end point of e, there exists some i so that $z > e$ in L_i. This means that we must find some M_i in which z is above both x and y in M_i. In fact we show that there is some i for which $z > x$ and $z > y$ in Q_i.

If z is an exterior vertex, say $z = u_i$, then z is the largest element in Q_i. Now suppose z is an interior vertex. Then for each $i = 1, 2, 3$, there is a path $S_i(z)$ from z to the ith exterior vertex u_i. The starting point of $S_i(z)$ is $v_0 = z$. If v_j has been determined, and v_j is an interior vertex, then v_{j+1} is the unique vertex so that the angles at v_j on either side of the edge $v_j v_{j+1}$ are colored $i+1$ and $i+2$.

The paths $S_1(z), S_2(z)$, and $S_3(z)$ are pairwise disjoint and partition T into three regions R_1, R_2, and R_3 as shown in fig. 10.5.

If the edge $e = xy$ joins two vertices in the region R_i, then z is greater than both x and y in Q_i. This completes the proof. □

It is well known that the problem of deciding whether a poset **P** satisfies $\dim(\mathbf{P}) \leqslant 2$ belongs to the class P of problems admitting a polynomial-time solution. For fixed $t \geqslant 3$, Yannakakis (1982) proved that the problem of deciding whether a poset **P** satisfies $\dim(\mathbf{P}) \leqslant t$ is NP-complete. For these reasons, Schnyder's theorem (10.3) is all the more striking since it equates a well-known polynomial-time problem, planarity testing, with an apparently NP-complete problem, deciding whether a particular poset has dimension at most 3. However, the poset being tested has a special form. The maximal elements all have degree two in the comparability graph. Also, it is not known whether it is NP-complete to answer whether the dimension of a height-two poset is at most 3. The answer is "yes" for dimension 4 or more.

Schnyder's theorem has been applied to find efficient algorithms for laying out a planar graph on a grid (see Kant 1992, Schnyder 1990, Schnyder and Trotter 1995). Recently, Brightwell and Trotter (1994a) have extended Schnyder's theorem to arbitrary planar maps.

Theorem 10.2. *Let* **G** *be a planar multigraph and let D be a drawing of* **G** *in*

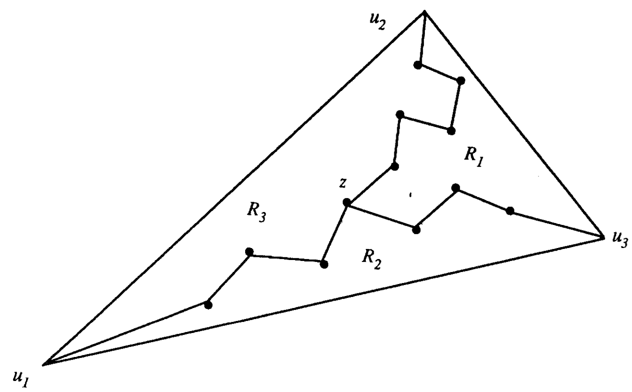

Figure 10.5.

the plane so that no edges cross. Then let $\mathbf{P}(D)$ *be the poset consisting of the vertices, edges and faces of the drawing D partially ordered by inclusion. Then* $\dim(\mathbf{P}(D)) \leqslant 4$.

The proof of Theorem 10.4 depends on the development of special graph-theoretic techniques applied to ordinary planar graphs satisfying a property somewhat weaker than 3-connectedness. The argument is inductive and required Brightwell and Trotter (1993) to first establish the following theorem.

Theorem 10.3. *Let* $\mathbf{M}$ *be a convex polytope in* $\mathbb{R}^3$, *and let* $\mathbf{P}_M$ *denote the poset consisting of the vertices, edges and faces of* $\mathbf{M}$ *partially ordered by inclusion. Then* $\dim(\mathbf{P}_M) = 4$.

In fact, the proof of Theorem 10.3 yields the even stronger conclusion that the subposet of $\mathbf{P}_M$ determined by the vertices and the faces of $\mathbf{M}$ is 4-irreducible. Theorem 10.3 cannot be extended to yield a bound of the dimension of the face lattice of a convex polytope in $\mathbb{R}^n$ for $n \geqslant 4$. This is due to the existence of cyclic polytopes [see the discussion in Brightwell and Trotter (1993)].

However, Theorem 10.3 can be extended to surfaces of higher genus since it is easy to prove by induction on n the existence of a function $f(n)$ so that the dimension of the poset of vertices, edges and faces of a multigraph drawn without edge crossings on a surface of genus n has order dimension at most $f(n)$. The only difficulty encountered in establishing the existence of $f(n)$ is the case $n = 0$. Here we know of no elementary proof of any finite bound, although of course Theorem 10.3 yields an upper bound of 4 in this case.

11. Regressions and monotone chains

If $\mathbf{P} = (X, P)$ is a poset, we call a map $f: X \to X$ a *regression* if $f(x) \leqslant x$ for every $x \in X$. When $C = \{x_1 < x_2 < \cdots < x_k\}$ is a k-element chain in $\mathbf{P}$, we say that a regression f is *monotone* on C if $f(x_1) \leqslant f(x_2) \leqslant \cdots \leqslant f(x_k)$. By convention, a regression is monotonic on any 1-element chain.

For $k \geqslant 1$, there are several interesting conditions on a poset which guarantee that every regression is monotonic on some k-element chain. Here is an important example due to Rado (1971).

Theorem 11.1. *For every $k \geqslant 1$, there exists an integer $n_0 = n_0(k)$ so that if $n \geqslant n_0$ and f is a regression on the subset lattice $\mathbf{2}^n$, then f is monotonic on some k-element chain.*

An alternative proof of Theorem 11.1 has been provided by Harzheim (1982) and this argument extends to a wider class of posets. However, neither argument gives much information about how large n_0 must be in terms of k. This is not surprising in view of the arguments' dependence on Ramsey-theoretic tools emphasizing existence.

By way of contrast, we present in this section a sharp result for posets of bounded width. The result is due to Peck et al. (1984).

Theorem 11.2. *Let w and k be positive integers and let $\mathbf{P} = (X, P)$ be a poset of width at most w. If $|X| \geqslant (w+1)^{k-1}$ then every regression is monotonic on some k-element chain.*

Proof. We proceed by induction on k, noting that the case $k = 1$ is trivial. Now assume $k \geqslant 2$ and that the theorem holds for smaller values of k. Let $\mathbf{P} = (X, P)$ be any poset of width at most w and let f be any regression on $\mathbf{P}$. We show f is monotonic on some k-element chain.

For each $x \in X$, let $H(x)$ be the largest t for which there is a t-element chain $x_1 < x_2 < \cdots < x_t = x$ on which f is monotonic. Without loss of generality $H(x) \leqslant k-1$ for all $x \in X$.

Then let $Y = \{x \in X\colon H(x) < k-1\}$, $F = \{x \in X\colon H(x) = k-1, f(x) = x\}$ and $M = \{x \in X\colon H(x) = k-1, f(x) \neq x\}$. Evidently, $X = Y \cup F \cup M$ is a partition.

Now suppose that F is not an antichain. Choose $x, x' \in F$ with $x < x'$ in P. Then choose a $(k-1)$-element chain $x_1 < x_2 < \cdots < x_{k-1} = x$ on which f is monotonic. Then adding x' to this chain yields a k-element chain on which f is monotonic since $f(x_1) \leqslant f(x_2) \leqslant \cdots \leqslant f(x_{k-1}) = f(x) = x < x' = f(x')$. The contradiction shows that F is an antichain and thus $|F| \leqslant w$.

Next suppose that $x \in M$ and that $H(f(x)) = k-1$. Then we may choose a $(k-1)$-element chain $x_1 < x_2 < \cdots < x_{k-1} = f(x)$ on which f is monotonic. Since $f(x) < x$, we may add x to this chain to obtain a k-element chain on which f is monotonic. The contradiction shows $H(f(x)) \leqslant k-2$ for every $x \in M$, i.e., $k \geqslant 3$ and $f(M) \subset Y$.

Now let $y \in Y$. It is easy to see that $f(y)$ also belongs to y. Thus the restriction of f to Y is a regression. Since this restriction is not monotonic on any chain of $k-1$ points, it follows from the inductive hypothesis that $|Y| < (w+1)^{k-2}$.

Since $|X| \geqslant (w+1)^{k-1}$, $|Y| < (w+1)^{k-2}$ and $|F| \leqslant w$, we conclude that

$$\begin{aligned} |M| &= |X| - |Y| - |F| \\ &> (w+1)^{k-1} - (w+1)^{k-2} - w \\ &= w[(w+1)^{k-2} - 1] \\ &\geqslant w|Y| \end{aligned}$$

Since the width of $\mathbf{P}$ is at most w, it follows that there is some $y_0 \in Y$ for which the inverse image $f^{-1}(y_0)$ is not an antichain. We may then choose distinct points $x, x' \in M$ for which $f(x) = f(x') = y_0$ and $x < x'$ in P. As before, we choose a $(k-1)$-element chain $x_1 < x_2 < \cdots < x_{k-1} = x$ on which f is monotonic and add x' to form the desired k-element chain. □

The reader may enjoy the challenge of showing that the inequality $|X| \geqslant (w+1)^{k-1}$ in Theorem 11.2 is best possible. The basic idea is to fix w and then construct a poset $\mathbf{P}_k = (X_k, P_k)$ and a regression f on $\mathbf{P}_k$ by induction on k. The poset $\mathbf{P}_1$ is a w-element antichain, and $\mathbf{P}_k$ is constructed by placing w disjoint chains, each containing $1 + (w+1)^{k-2}$ points, on top of $\mathbf{P}_{k-1}$. We refer the reader to Peck et al. (1984) for further details.

There appears to be some intrinsic connection between regressions and arithmetic progressions. Following Trotter and Winkler (1987), we define an *arithmetic progression* in a poset $\mathbf{P} = (X, P)$ as a chain $x_1 < x_2 < \cdots < x_t$ for which there is a constant d so that there are exactly d points in each of the intervals $\{y \in X\colon x_i \leqslant y < x_{i+1}\}$ for $i = 1, 2, \ldots, t-1$. The following result is due to Trotter and Winkler (1987).

Theorem 11.3. *Let k and w be positive integers and let $\varepsilon > 0$. Then there exists a number $n_0 = n_0(k, w, \varepsilon)$ so that if $\mathbf{P} = (X, P)$ is a poset of width at most w and $|X| \geqslant n_0$, then for every subset $S \subset X$ with $|S| > \varepsilon|X|$, there is a k-element chain $x_1 < x_2 < \cdots < x_k$ contained in S which is also a k-term arithmetic progression in* $\mathbf{P}$.

The proof of Theorem 11.3 proceeds by induction on w with the case $w = 1$ being a restatement of Szemerédi's (1975) celebrated theorem on arithmetic progressions in subsets of $\mathbb{N}$ having positive upper density. It is reasonable to conjecture that for each $k \geqslant 1$, there is some $n_0 = n_0(k)$ so that if $\mathbf{L} = (X, P)$ is any distributive lattice with $|X| \geqslant n_0$, then every regression on $\mathbf{L}$ is monotonic on some k-element chain. This is supported by Theorems 11.1 and 11.2.

Also, we believe that for every $k \geqslant 1$ and every $\varepsilon > 0$, there is some $n_0 = n_0(k, \varepsilon)$ so that if $\mathbf{L} = (X, P)$ is a distributive lattice with $|X| \geqslant n_0$ and S is any subset of X with $|S| > \varepsilon|X|$, then S contains a k-term arithmetic progression. It is an easy exercise to show that this conjecture holds in the case where $\mathbf{L}$ is a subset lattice of the form $\mathbf{2}^n$.

Some modest progress has been made on these conjectures. Alon et al. (1987) study regressions on up sets in $\mathbf{n}^2$, while Kahn and Saks (1988) show that for each $\varepsilon > 0$, there exists an integer n_0 so that if $\mathbf{L} = (X, P)$ is a distributive lattice and $|X| \geqslant n_0$, then any antichain in $\mathbf{L}$ has less than $\varepsilon|X|$ points.

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