Primal dual algorithms for UFL

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JMS algorithm

$U := C,\ \alpha_i = 0 \quad \forall_i$

$$offer(j) = \begin{cases} max(\alpha_j - d_{ij}, 0) & j \text{ is unconnected} \\ max(d_{i'j} - d_{ij}, 0) & j \text{ is connected with } i' \end{cases}$$

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While $U \neq \emptyset$, for every client $j \in U$, increase the parameter α_j at the same rate, until one of the following events occurs

- For some unopened facility i, the total offer that it receives from clients is equal to the cost of opening i. In this case, we open facility i, and for every client j (connected or unconnected) which has a nonzero offer to i, we connect j to i and remove j from U.
- For some unconnected client j, and some open facility i, $\alpha_j = d_{ij}$. In this case, connect client j to facility i and remove j from U.

JMS algorithm - the factor revealing LP

$$\max \frac{\sum_{i=1}^{k} \alpha_i}{f + \sum_{i=1}^{k} d_i}$$

$$\alpha_i \leq \alpha_{i+1} \qquad \forall_{i<k}$$

$$r_{j,i} \geq r_{j,i+1} \qquad \forall_{j \leq i<k}$$

$$\alpha_i \leq r_{j,i} + d_i + d_j \qquad \forall_{j<i \leq k}$$

$$\sum_{j=1}^{i-1} \max\{r_{j,i} - d_j, 0\} + \sum_{j=i}^{k} \max\{\alpha_i - d_j, 0\} \leq f \qquad \forall_{i \leq k}$$

$$\alpha_i, r_{j,i}, d_i, f \geq 0 \qquad \forall_{j \leq i \leq k}$$

JMS algorithm - the (bi-factor) factor revealing LP

$$z_k = max \frac{\sum_{i=1}^{k} \alpha_i - \lambda_f \cdot f}{\sum_{i=1}^{k} d_i}$$

$$\alpha_i \leq \alpha_{i+1} \qquad \forall_{i<k}$$

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JMS algorithm - the upper bound

$$y_m = max \; \frac{\sum_{i=1}^{m} \alpha'_i - \lambda_f \cdot f}{\sum_{i=1}^{m} d'_i}$$

$$\alpha'_i \leq \alpha'_{i+1} \qquad \forall_{i<m}$$

$$r'_{j,i} \geq r'_{j,i+1} \qquad \forall_{j \leq i < m}$$

$$\alpha'_i \leq r'_{j,i-1} + d'_i + d'_j \qquad \forall_{j<i-1 \;\wedge\; i \leq m}$$

$$r'_{i,i} \leq \alpha'_i \qquad \forall_{i \leq k}$$

$$\sum_{j=1}^{i} max\{r'_{j,i} - d'_j, 0\} + \sum_{j=i+1}^{m} max\{\alpha'_i - d'_j, 0\} \leq f \qquad \forall_{i \leq m}$$

$$\alpha'_i, r'_{j,i}, d'_i, f \geq 0 \qquad \forall_{j \leq i \leq m}$$

JMS - bifactor analysis

Lemma

Value of z_{k_1} is less or equal to z_{k_2}, where $k_2 = k_1 \cdot l$ and $l \in \mathbb{N} \setminus \{0\}$, and $\lambda_f \geq 1$.

Lemma

Let $m, k \in \mathbb{N} \setminus \{0\}$, $\lambda_f \geq 1$. The value of the objective function y_m (the upper-bound) is greater than or equal to the value of the objective function z_k (factor revealing LP)

Theorem

JMS'(δ) is a $(1, y_m)$-approximation algorithm for UFL, for any $m \in \mathbb{N} \setminus \{0\}$.

$y_m \leq 1.95238219$ for $\delta = 1.023$ and $m = 250$

Corollary

k-median problems

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i \in \mathcal{F}, j \in \mathcal{C}} d(i,j) x_{ij} & & \\
\text{such that} \quad \sum_{i \in \mathcal{F}} y_i & \leq k & & \\
\sum_{i \in \mathcal{F}} x_{ij} & = 1 & & \text{for each } j \in \mathcal{C} \\
x_{ij} & \leq y_i & & \text{for each } i \in \mathcal{F}, j \in \mathcal{C} \\
x_{ij}, y_i & \geq 0 & & \text{for each } i \in \mathcal{F}, j \in \mathcal{C};
\end{aligned}
$$

Related work k-median

Selected previous resutls for k-Median

- $1 + \frac{2}{e}$ (hardness) [Guha, Khuller - SODA'98]
- $3 + \epsilon$ - approximation [Arya, Garg, Khandekar, Meyerson, Munagala, Pandit - STOC'01]
- 3.25 - approximation [Charikar, Li - ICALP'12]
- $(1 + \sqrt{3} + \epsilon) \simeq$ 2.7321-apx. [Li, Svensson - STOC'13]

our contribution

- 2.611-approximation for k-median

Bi-point solutions

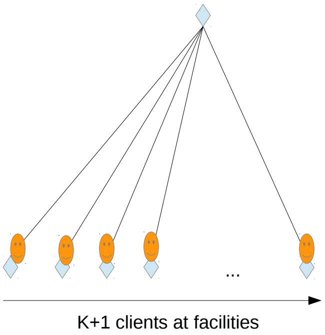
Let $c(F')$ be the optimal assignment cost if facilities in F' are open

Definition

if $\alpha|F_1| + \beta|F_2| \leq k$ for some $\alpha, \beta \geq 1, \alpha + \beta = 1$, then $(F_1, F_2, \alpha, \beta)$ is a bi-point solution of cost $\alpha c(F_1) + \beta c(F_2)$

Overview of the Li, Svensson algorithm

1. Obtain a bi-point solution of cost $2OPT$.
2. Round the bi-point solution to a pseudo solution using $k+c$ facilities for some constant c, loosing a factor of $1.366+\epsilon$.
3. Reduce approximation to pseudo-approximation.



Getting bi-point from bi-factor

Definition

An algorithm for UFL is a (λ_f, λ_c)-approximation algorithm if it delivers solutions of cost $\lambda_f F^* + \lambda_c C^*$.

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Jain, Mahdian, and Saberi gave a $(1, 2)$ approximation algorithm for UFL, which was used to get the bi-point solution in the Li, Svensson algorithm for k-median.
Next, I will tell you how to get a $(1, 1.9524)$ algorithm for UFL.

JMS'(δ) algorithm

$U := C,\ \alpha_i = 0 \quad \forall_i$

$$offer(j) = \begin{cases} \delta \cdot max(\alpha_j - d_{ij}, 0) & j \text{ is unconnected} \\ max(d_{i'j} - d_{ij}, 0) & j \text{ is connected with } i' \end{cases}$$

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JMS'(δ) algorithm - the factor revealing LP

$$max \ \frac{\sum_{i=1}^{k} \delta \cdot (\alpha_i - r_{i,i}) + r_{i,i}}{f + \sum_{i=1}^{k} d_i}$$

$$\alpha_i \leq \alpha_{i+1} \qquad \forall_{i<k}$$

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$$\sum_{j=1}^{i-1} max\{r_{j,i} - d_j, 0\} + \delta \cdot \sum_{j=i}^{k} max\{\alpha_i - d_j, 0\} \leq f \qquad \forall_{i \leq k}$$

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$$z_k = max \frac{\sum_{i=1}^{k} \delta \cdot (\alpha_i - r_{i,i}) + r_{i,i} - f}{\sum_{i=1}^{k} d_i}$$

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JMS'(δ) - analysis

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Corollary

JMS'($\delta = 1.023$) is is at most $(1, 1.9524)$-apx for UFL problem.

Concluding the k-median part

for the k-median we got

- 2 -> 1.9524 for getting bi-point,
- $1.366 + \epsilon$ -> $1.3371 + \epsilon$ for founding bi-point to pseudo-solution,
- $O(\frac{1}{\epsilon^2})$ extra facilities -> $O(\frac{1}{\epsilon} log \frac{1}{\epsilon})$ extra facilities,
- $2.732 + \epsilon$ -> $2.611 + \epsilon$-approximation for k-median.

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Let us now forget about k-median and consider capacitated k-median.